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Integrable Stochastic Dynamics and Gelfand-Tsetlin patterns

by

Theodoros Assiotis

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Contents

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Declarations

I declare that, to the best of my knowledge, the material in this thesis is original and my own work, conducted under the supervision of Jon Warren, unless otherwise indicated.

The mathematical content is taken from 5 research papers [6], [7], [8], [9], [10], one of which [6] is joint work with Jon Warren and Neil O'Connell. More precisely:

1. Chapter 1 is a condensed version of [6], which is joint work with Jon Warren and Neil O'Connell and which has been submitted for publication. Some of the longer proofs are omitted although the strategy and key ingredients are explained in detail.

2. Chapter 2 is based on an edited version of [10] which has been accepted for publication at Journal of Theoretical Probability.

3. Chapter 3 is based on an edited version of [7] which has been submitted for publication.

4. Chapter 4 is based on an edited version of [9] which has been published at Electronic Communications in Probability.

5. Chapter 5 is based on an edited version of [8] which has been submitted for publication.

The material in this thesis has not been submitted for any other degree either at the University of Warwick or any other University.

Abstract

In this thesis we study several topics in Probability Theory and Mathematical Physics. These include interacting particle systems, random matrices, models of stochastic surface growth and branching graphs which are closely related to representation theory. The main common thread is that of stochastic dynamics with certain underlying integrable structure, which allows for many exact computations and explicit formulae.

Chapter 0

Introduction

This PhD thesis covers several topics in Probability Theory and Mathematical Physics, more specifically interacting particle systems, random matrix theory, stochastic surface growth and the study of branching graphs which are also closely related to asymptotic representation theory. The main common theme is that of stochastic dynamics with certain underlying integrable structure which allows for many exact computations and explicit formulae. With these formulae at one's disposal, very strong limiting statements as the size of the system grows can be obtained. An attempt that was made throughout this thesis was to understand at the right degree of generality the rigid algebraic structures underlying these various models. This approach not only sheds light on the origin of their exact solvability but also streamlines many of the computations of certain special cases.

We will now describe the contents and results of each chapter informally, surveying in the process related work which we build upon or generalize. We will not attempt though to give a general overview of the rapidly growing area of integrable probabilistic systems as that would be doomed to fail, the reader can consult the excellent surveys [32], [22].

Before continuing we remark that we consider the two main results of this thesis to be the following. Firstly, the construction of an infinite dimensional Feller-Markov process preserving the Hua-Pickrell measures on Hermitian matrices via a novel approach, this appears as Theorem 0.4 below. We note that in the random matrix setting a construction of such an infinite-dimensional Feller process was not achieved before this work. The second main result is the explicit computation of the distribution of an inhomogeneous random growth and decay process of a 2-dimensional surface, in Theorem 0.8 below. This is given in terms of a determinantal correlation kernel involving one dimensional orthogonal polynomials and their orthogonality measures.

0.1 Chapter 1: Interlacing diffusions

Diffusive (2+1)-dimensional dynamics Interlacing arrays, also called Gelfand-Tsetlin patterns, appear in diverse areas of Mathematics. These can be viewed as two dimensional configurations of particles (see the figure below), that live in either subsets of $\mathbb{R}$ or $\mathbb{Z}$, and which satisfy some constraints to be made precise below. Their continuous incarnations arise in matrix theory, by Rayleigh's theorem, if one looks at eigenvalues of consecutive minors of Hermitian matrices, a fact of fundamental importance in Chapter 3. They also come up in representation theory, the original motivation behind Gelfand's and Tsetlin's study, and in branching graphs as we shall explain in Chapter 5.

We will be interested in models where the particles evolve in time in a random fashion. The study of such stochastic dynamics in interlacing arrays has seen a great deal of activity in the past two decades. Arguably there are two sources of integrable dynamics in such arrays. The first one, is based on the combinatorial algorithm of the RSK correspondence. This was first applied in this setting by Johansson [86] and Baik-Deift-Johansson [11]. The dynamical perspective was then studied and greatly developed by O'Connell [111], [112], Biane-Bougerol-O'Connell [17], Corwin-O'Connell-Seppalainen-Zygouras [52], elucidating diverse connections from the Littelman path model to Whittaker functions and random polymers.

The other source is the so called *push-block* dynamics, introduced by Borodin and Ferrari [21] in the discrete setting based an idea of Diaconis and Fill [57], see also [166]. This approach further developed by Borodin, Olshanski, Corwin, Petrov among others in [28], [19], [33], [34] for example. It is an interesting fact the continuum analogue of these dynamics predates [21] and originates with the work of Warren [164] that we will explain in detail later on. It is this kind of dynamics that will be investigating in this thesis.

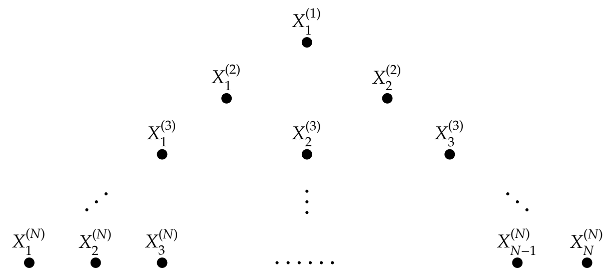
Although such constructions on interlacing arrays are interesting in their own right, they also allow one to prove results such as the following, on Brownian percolation, first proven by Baryshnikov [14] and Gravner-Tracy-Widom [79]:

$$\sup_{0=t_0\le t_1\le\cdots\le t_N=1}\sum_{i=1}^{N}\left(\beta_i(t_i)-\beta_i(t_{i-1})\right)\stackrel{d}{=}\lambda_N$$

where the β_i are independent standard Brownian motions and λ_N is the largest eigenvalue of a N-dimensional GUE random matrix. Part of the original motivation for the project was to extend such a result to the other two classical unitarily invariant random matrix ensembles, the Laguerre (LUE) and Jacobi (JUE).

We now go on to describe the results obtained more precisely. We will denote throughout this chapter by $W^N=\{(x_1,\cdots,x_N):x_1\le\cdots\le x_N\}$ the continuum Weyl chamber. We will say that $y\in W^N$ interlaces with $x\in W^{N+1}$ and denote this by $y\prec x$ if the following inequalities are satisfied: $x_1\le y_1\le\cdots\le y_N\le x_{N+1}$. We will write $\mathbb{GT}_c(N)$ for a continuum Gelfand-Tsetlin pattern:

$$\mathbb{GT}_c(N)=\{(X^{(1)},\cdots,X^{(N)}):X^{(n)}\in W^n,\ X^{(n)}\prec X^{(n+1)}\},$$



Gelfand-Tsetlin patterns can also be mapped to a height function H, as illustrated in the figure in the introduction for Chapter 5 below for a discrete variant.

The aim of this chapter is to introduce a class of diffusive stochastic dynamics on interlacing arrays with integrable structure. The evolution can be described informally as follows: On the k^{th} level of the array the particles evolve as independent copies of a one-dimensional diffusion process, with state space an interval I with endpoints l and r, *reflected off* the paths of the particles at the $(k-1)^{th}$ level. The rigorous definition is through SDEs with reflection described below:

$$d\mathbb{X}_i^{(n)}(t) = \sqrt{2a_n\left(\mathbb{X}_i^{(n)}(t)\right)}d\beta_i^{(n)}(t) + b_n\left(\mathbb{X}_i^{(n)}(t)\right)dt + dK_i^{(n),-} - dK_i^{(n),+}, \tag{1}$$

driven by an array $\left(\beta_i^{(n)}(t); t \geq 0, 1 \leq i \leq n \leq N\right)$ of $\frac{N(N+1)}{2}$ independent standard Brownian motions. The positive finite variation processes $K_i^{(n),-}$ and $K_i^{(n),+}$ are such that $K_i^{(n),-}$ increases only when $\mathbb{X}_i^{(n)} = \mathbb{X}_{i-1}^{(n-1)}$, $K_i^{(n),+}$ increases only when $\mathbb{X}_i^{(n)} = \mathbb{X}_i^{(n-1)}$ with $K_1^{(N),-}$ increasing when $\mathbb{X}_1^{(N)} = l$ and $K_N^{(N),+}$ increasing when $\mathbb{X}_N^{(N)} = r$, so that $\mathbb{X} = \left(\mathbb{X}^{(1)}, \cdots, \mathbb{X}^{(N)}\right)$ stays in $\mathbb{GT}_c(N)$ forever.

Surprisingly, to each level n of the pattern one can associate a Markovian semigroup $\mathfrak{P}^{(n)}(t)$. This is unexpected since if one looks at the evolution of level n in the SDEs then it is clearly non-autonomous, the interaction though being only through the local finite variation terms $K_i^{(n),-}$ and $K_i^{(n),+}$. Also, to any pair of consecutive levels $n, n+1$ a Markov kernel $\mathfrak{L}_n^{n+1}(x^{(n+1)}, dx^{(n)})$ is associated describing the conditional distribution of level n given that the configuration of particles at level $n+1$ is $x^{(n+1)}$. All of these kernels are given as ratios and products of determinants. Then we have the following informal statement (of some) of the main results of this chapter (see Section 1.3 and Proposition 1.17 for precise statements):

Main results Under **(A)** certain consistency assumptions on the coefficients and **(B)** certain technical conditions for well-posedness of the SDEs we consider the process $(\mathbb{X}(t); t \geq 0) = \left(\left(\mathbb{X}^{(1)}(t), \cdots, \mathbb{X}^{(N)}(t)\right); t \geq 0\right)$ in $\mathbb{GT}_c(N)$ satisfying the SDEs (1) and initialized according to a measure of the form below that we call Gibbs,

$$\nu_N(dx^{(N)})\mathfrak{L}_{N-1}^N(x^{(N)}, dx^{(N-1)}) \cdots \mathfrak{L}_1^2(x^{(2)}, dx^{(1)}). \tag{2}$$

Then, $\left(\mathbb{X}^{(n)}(t); t \geq 0\right)$ for $1 \leq n \leq N$ is distributed as a Markov process with transition semigroup $\mathfrak{P}^{(n)}(t)$ and for fixed $T > 0$ the law of $\left(\mathbb{X}^{(1)}(T), \cdots, \mathbb{X}^{(N)}(T)\right)$ is given by,

$$\left(\nu_N \mathfrak{P}_T^{(N)}\right)(dx^{(N)}) \mathfrak{L}_{N-1}^N(x^{(N)}, dx^{(N-1)}) \cdots \mathfrak{L}_1^2(x^{(2)}, dx^{(1)}). \tag{3}$$

A special case of this result when $a_n(x) \equiv \frac{1}{2}, b_n(x) \equiv 0$, i.e. when the process consists of standard Brownian motions reflected off each other is the so called Warren process, introduced in [164]. Then in this case,

$$\mathfrak{P}^{(n)}(t)(x, dy) = \frac{\Delta_N(y)}{\Delta_N(x)} \det\left(\phi_t(y_j - x_i)\right)_{i,j=1}^n dy_1 \cdots dy_n$$

where $\Delta_n(x) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$ is the Vandermonde determinant and $\phi_t(z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}$ is the heat kernel. This is the transition kernel of Dyson's Brownian motion:

$$X_i(t) = x_i + \beta_i(t) + \sum_{j \neq i}^n \int_0^t \frac{1}{X_i(s) - X_j(s)} ds$$

where the β_i are independent standard Brownian particles. This diffusion can also be interpreted as n independent Brownian motions conditioned by a Doob's h-transform never to intersect. Moreover, the Markov kernels $\mathfrak{L}_{n-1}^n(x, dy)$ are given (explicitly for x in the interior of W^n) by:

$$\mathfrak{L}_{n-1}^n(x, dy) = \frac{(n-1)! \Delta_{n-1}(y)}{\Delta_n(x)} \mathbf{1}(y \prec x) \, dy_1 \cdots dy_{n-1}. \tag{4}$$

We will sometimes refer to these kernels as the Vandermonde links. Then the distribution at time T of $\left(\mathbb{X}^{(1)}(T), \cdots, \mathbb{X}^{(N)}(T)\right)$ if started from the origin is given by the GUE minor/corners process with diffusivity T, see [164].

To explain the general result further we need to introduce the notion of Siegmund duality for one-dimensional diffusions. The various incarnations of this duality play an important role throughout this thesis, in particular in the last chapter. Let $X(t)$ be a one-dimensional diffusion process, with state space an interval I with endpoints l and r and with generator,

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}.$$

Let $m(y)dy$ be the measure with respect to which this is reversible and $p_t(x, y)$ its transition density with respect to Lebesgue measure in (l, r). Then, we define its (Siegmund) dual diffusion $\hat{X}(t)$ with generator $\hat{L}$:

$$\hat{L} = a(x)\frac{d^2}{dx^2} + \left(a'(x) - b(x)\right)\frac{d}{dx}$$

and with dual boundary conditions at l and r (for example if $X(t)$ is reflecting at l then $\hat{X}(t)$ is absorbing at l, for the rigorous definition, see subsection 1.2.1). Similarly, denote by $\hat{m}(y)dy$ the measure with respect to which $\hat{X}(t)$ is reversible and $\hat{p}_t(x,y)$ its transition density in (l,r). For example standard Brownian motion is self-dual. Take copies of these dual diffusions defined on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$. Then, the key property they satisfy is:

$$\mathbb{P}_x\left(X(t) \le y\right) = \mathbb{P}_y\left(\hat{X}(t) \ge x\right), \forall x, y \in (l,r), t \ge 0.$$

Consider $\mathcal{P}_N(t)$ to be the sub-Markov Karlin-McGregor semigroup associated to N independent of L-diffusions with transition kernel:

$$\mathcal{P}_N(t)(x,dy) = \det\left(p_t(x_i,y_j)\right)_{i,j=1}^N dy_1 \cdots dy_N.$$

The probabilistic interpretation of this is the following. Let $\mathbb{P}^{\otimes N}_{(x_1,\cdots,x_N)}$ be the law of N independent copies of N one-dimensional diffusions with generator L started at $(x_1,\cdots,x_N) \in W^N$. If $\tau = \inf\{s \ge 0 : \exists i \neq j \text{ s.t. } X_i(t) = X_j(t)\}$ is the first collision time and $\mathcal{A} \subseteq W^N$ a Borel set:

$$\mathbb{P}^{\otimes N}_{(x_1,\cdots,x_N)}\left[(X_1(t),\cdots,X_N(t)) \in \mathcal{A}, \tau > t\right] = \int_{\mathcal{A}} \det\left(p_t(x_i,y_j)\right)_{i,j=1}^N dy_1 \cdots dy_N.$$

We let $\hat{\mathcal{P}}_N(t)$ be the analogous semigroup associated to $\hat{L}$-diffusions and also consider the following positive kernel, for $x \in W^{N+1}, y \in W^N$:

$$\Lambda_{N,N-1}(x,dy) = \prod_{i=1}^{N-1} \hat{m}(y_i)\mathbf{1}\left(y \prec x\right) dy_1 \cdots dy_{N-1}.$$

Then, subject to boundary conditions at l and r we have the intertwining relation between Karlin-McGregor semigroups:

$$\mathcal{P}_N(t)\Lambda_{N,N-1} = \Lambda_{N,N-1}\hat{\mathcal{P}}_{N-1}(t), \forall t \ge 0. \tag{5}$$

This is the key ingredient in constructing consistent dynamics for interlacing arrays. Analogous intertwining relations exist for N and N particles, see subsection 1.2.4.

One could then consider the Doob-transformed version, by a strictly positive eigenfunction, of these semigroups, under which the paths never intersect. With $\hat{h}$ being a strictly positive eigenfuction of $\hat{\mathcal{P}}_{N-1}(t)$ with eigenvalue c then from (5) we get that $h(x) = \left[\Lambda_{N,N-1}\hat{h}\right](x)$ is a strictly positive eigenfunction of $\mathcal{P}_N(t)$ and the transformed semigroup is given by:

$$\mathcal{P}^h_N(t)(x,dy) = e^{-ct}\frac{h(y)}{h(x)}\det\left(p_t(x_i,y_j)\right)_{i,j=1}^N dy_1 \cdots dy_N.$$

Also, define:

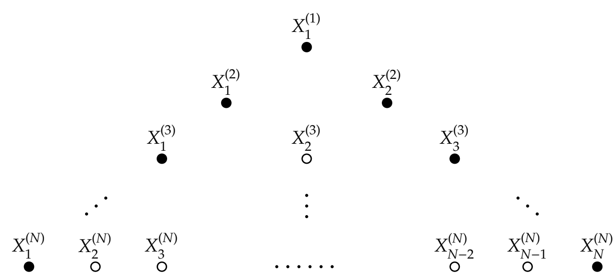
$$\Lambda^{\hat{h}}_{N,N-1}(x,dy) = \frac{\hat{h}(y)\prod_{i=1}^{N-1}\hat{m}(y_i)\mathbf{1}(y \prec x)}{h(x)}dy_1\cdots dy_{N-1}.$$

It is then a trivial exercise using (5) that:

$$\mathcal{P}^h_N(t)\Lambda^{\hat{h}}_{N,N-1} = \Lambda^{\hat{h}}_{N,N-1}\hat{\mathcal{P}}^{\hat{h}}_{N-1}(t), \forall t \geq 0. \tag{6}$$

These are the sort of semigroups that can be associated to each level of the array, giving rise to non-intersecting paths. More precisely, in the notation of the main result $\mathfrak{P}^{(n)}(t)$ is a Doob's h-trasformed Karlin McGregor semigroup $\mathcal{P}^h_N(t)$ for some one dimensional diffusion with generator L and a strictly positive eigenfunction h. Similarly, the Markov kernel $\mathfrak{L}^N_{N-1}$ is equal to $\Lambda^{\hat{h}}_{N,N-1}$.

Particle systems with one-sided collisions Another model studied in detail in this chapter is the particle systems with one-sided collisions appearing at the edge of the (continuous) Gelfand-Tsetlin valued processes we consider. In the figure below, the particles we will be concerned with are denoted in $\bullet$.



We consider the particles on the right edge of the pattern, $\left(X_1^{(1)}(t),\cdots,X_N^{(N)}(t);t\geq 0\right)$. An important observation is that this particle system is autonomous and thus (modulo well-posedness of the SDEs below) its evolution Markovian. In the following equations the γ_i^i are independent standard Brownian motions and $K_i^{i,-}$ are positive finite variation processes with the measure $dK_i^{i,-}$ supported on $\left\{t : X_i^{(i)}(t) = X_{i-1}^{(i-1)}(t)\right\}$ (these terms correspond to one-sided reflection):

$$\begin{aligned} dX_1^{(1)}(t) &= \sqrt{2a(X_1^{(1)}(t))}d\gamma_1^1(t) + b^{(1)}(X_1^{(1)}(t))dt, \\ &\vdots \\ dX_m^{(m)}(t) &= \sqrt{2a(X_m^{(m)}(t))}d\gamma_m^m(t) + b^{(m)}(X_m^{(m)}(t))dt + dK_m^{m,-}(t), \\ &\vdots \\ dX_N^{(N)}(t) &= \sqrt{2a(X_N^{(N)}(t))}d\gamma_N^N(t) + b^{(N)}(X_N^{(N)}(t))dt + dK_N^{N,-}(t). \end{aligned} \tag{7}$$

In order for this system to be exactly solvable we assume $a(x)$ and $b^{(k)}$ are of the form:

$$a(x) = a_0 + a_1 x + a_2 x^2 \ , \ b(x) = b_0 + b_1 x,$$
$$b^{(k)}(x) = b(x) + (N-k)a'(x) = b_0 + (N-k)a_1 + (b_1 + 2(N-k)a_2)x.$$

For $a(x) \equiv \frac{1}{2}, b(x) \equiv 0$ we obtain Brownian motions with one-sided collisions. This well-studied model is also known Brownian directed percolation, as in the result of Baryshnikov referenced above. We now briefly explain how to go from the particle system to the percolation interpretation. The equations in this case are as follows:

$$X_N^N(t) = \beta_N^N(t) + K_N^{N,-}(t).$$

But, by Skorokhod's lemma, see [134], $K_N^{N,-}$ is actually given explicitly:

$$K_N^{N,-}(t) = \sup_{s \le t} \left(X_{N-1}^{N-1}(s) - \beta_N^N(s) \right)$$

and iterating this we obtain, which along with the main result mentioned before gives Baryshnikov's theorem:

$$X_N^N(t) = \sup_{0=t_0 \le t_1 \le \cdots \le t_N = t} \sum_{i=1}^{N} \left(\beta_i^i(t_i) - \beta_i^i(t_{i-1}) \right).$$

We shall also assume that the endpoints l and r of the state space are unattainable from the interior of the interval I°. In one-dimensional diffusion jargon l and r are both either entrance or natural boundary points; there are necessary and sufficient conditions on $a(x)$ and $b(x)$ for this to be the case, that can be found in the appendix of [6] for example.

We will denote by $p_t^{(k)}(x,y)dy$ the transition kernel of the one-dimensional diffusion process given by the strong solution to the (unconstrained) *SDE*:

$$dX(t) = \sqrt{2a(X(t))}d\gamma(t) + b^{(k)}(X(t))dt.$$

Then, we have a Schutz type formula [141] for the transition density of this particle system, see Section 1.4 and Proposition 1.18 in particular:

Theorem 0.1. *Assume $a(x) = a_0 + a_1 x + a_2 x^2$ and $b^{(k)}(x) = b_0 + (N-k)a_1 + (b_1 + 2(N-k)a_2)x$ and that the boundary points l and r are unattainable from the interior of the state space. Let $\mathbb{Q}_{(x_1,\cdots,x_N)}$ denote the law of the solution to the system of SDEs (7) starting from $(x_1,\cdots,x_N) \in W^N$. Then, for any $t>0$ and Borel subset $\mathcal{A}$ of W^N we have:*

$$\mathbb{Q}_{(x_1,\cdots,x_N)}\left(\left(X_1^{(1)}(t), \cdots, X_N^{(N)}(t) \right) \in \mathcal{A} \right) = \int_{\mathcal{A}} \det \left(\mathcal{S}_t^{(i),i-j}(x_i, y_j) \right)_{i,j=1}^{N} dy_1 \cdots dy_N$$

where,

$$\mathcal{S}_t^{(k),j}(x,x') = \begin{cases} \int_l^{x'} \frac{(x'-z)^{j-1}}{(j-1)!} p_t^{(k)}(x,z)dz, & j \geq 1 \\ \partial_{x'}^{-j} p_t^{(k)}(x,x'), & j \leq 0 \end{cases}.$$

A similar formula exists for the particle system at the left edge, see Proposition 1.19.

0.2 Chapter 2: Consistent dynamics for β-ensembles

The aim of this chapter is to study properties of dynamics for general β-ensembles, namely log-gases of the form:

$$\text{const} \times \prod_{1 \leq i < j \leq n} |x_j - x_i|^\beta \prod_{i=1}^n \mathcal{W}(x)dx$$

for some weight $\mathcal{W}(x)dx$.

The introduction of Gaussian β-ensembles, namely with $\mathcal{W}(x) = e^{-\frac{\beta}{2}x^2}$, can be traced back to the paper of Dyson [63] on the "threefold way" where he considered ensembles invariant under conjugation by the orthogonal, unitary and symplectic groups which correspond to $\beta = 1, 2$ and 4 respectively. Since then β-ensembles have been intensively studied and many names should be mentioned here but we just record some milestones. For example the famous Selberg integral (see Chapter 4 in [71]) gives the normalization constant for the β-ensemble with the Jacobi weight. Also, Johansson's tour de force contribution [86] in the study of Gaussian fluctuations for these log-gases introduced the general method of loop equations to attack global fluctuations questions. In another direction, Zirnbauer [172] generalizing the work of Dyson produced a general classification called the "tenfold way" based on symmetric spaces. More recently, Dumitriu and Edelman in a breakthrough paper [62] provided tridiagonal models for β-ensembles, which led to many developments, in particular the introduction of the general β Airy [132], Sine [155] and Bessel [133] processes using random operators by Virag, Valko and Rider among others.

Dyson was also the first in [64] to study dynamics for these β-ensembles, namely the β-Dyson Brownian motion:

$$dX_i^{(n)}(t) = dW_i^{(n)}(t) + \frac{\beta}{2} \sum_{j \neq i} \frac{1}{X_i^{(n)}(t) - X_j^{(n)}(t)} dt$$

where the $dW_i^{(n)}$ are independent standard Brownian motions. The study of this stochastic process, in particular the time to equilibrium of its stationary version, was the fundamental tool in Erdos and Yau's approach to universality for random matrices, see [67].

In the last few years a multilevel process for $\beta > 2$ was introduced by Gorin and Shkolnikov [75] taking values in a Gelfand-Tsetlin pattern such that the projection on level

k is given by k particle Dyson Brownian motion:

$$dX_i^{(n)}(t) = dW_i^{(n)}(t) + \sum_{j=1}^{n-1} \frac{\left(\frac{\beta}{2}-1\right)}{X_i^{(n)}(t) - X_j^{(n-1)}(t)} dt - \sum_{i\neq j} \frac{\left(\frac{\beta}{2}-1\right)}{X_i^{(n)}(t) - X_j^{(n)}(t)} dt.$$

Observe that interactions are more complicated than in the processes considered in Chapter 1, which correspond to $\beta = 2$, with now long range repulsions among particles taking place. The case $\beta = 2$ can be viewed as a singular case where the long range repulsion disappears and local time interactions appear.

In a follow-up paper [76] Gorin and Shkolnikov studied the particle system at the edge of this pattern and observed a novel decoupling phenomenon in a limit.

Their original construction goes via discrete approximation and a key ingredient is an intertwining relation between general β-Dyson Brownian motions of different dimensions, with links given by the Dixon-Anderson kernel introduced shortly. More recently, Ramanan and Shkolnikov [131] gave a different, directly in the continuum, approach to obtain this intertwining.

In this chapter, we perform the analogous task for the other two classical ensembles, the Laguerre with $\mathcal{W}(x) = x^{\frac{\beta}{2}\alpha - 1} e^{-\frac{\beta}{2}x}$ and Jacobi with $\mathcal{W}(x) = x^{\frac{\beta}{2}a-1}(1-x)^{\frac{\beta}{2}b-1}$. First, we need to introduce the relevant stochastic processes. Consider the unique strong solution to the system of SDEs:

$$dX_i^{(n)}(t) = 2\sqrt{X_i^{(n)}(t)}dB_i^{(n)}(t) + \beta\left(\frac{d}{2} + \sum_{1\leq j\leq n, j\neq i} \frac{2X_i^{(n)}(t)}{X_i^{(n)}(t) - X_j^{(n)}(t)}\right)dt, \tag{8}$$

where the $B_i^{(n)}$, $i = 1, \cdots, n$, are independent standard Brownian motions. This process, was introduced and studied by Demni in [55] in relation to Dunkl processes, (see for example [138]), namely stochastic evolutions that have the Dunkl differential and reflection operators as generators. These operators were first introduced by Dunkl in 1989 and have been fundamental in harmonic analysis on homogeneous spaces and the study of double affine Hecke algebras, playing a key role in the proof by Cherednik of the Macdonald conjectures, see [80]. We will call the solution of (8) the β-Laguerre process, since its distribution at time 1, if started from the origin, is given by the β-Laguerre ensemble, see [55].

We also consider the following evolution:

$$dX_i^{(n)}(t) = 2\sqrt{X_i^{(n)}(t)(1-X_i^{(n)}(t))}dB_i^{(n)}(t) + \beta\left(a - (a+b)X_i^{(n)}(t) + \sum_{1\leq j\leq n, j\neq i} \frac{2X_i^{(n)}(t)(1-X_i^{(n)}(t))}{X_i^{(n)}(t) - X_j^{(n)}(t)}\right)dt, \tag{9}$$

where, again, the $B_i^{(n)}, i = 1, \cdots, n$, are independent standard Brownian motions. It was first introduced and studied in [54] as a generalization of the eigenvalue evolutions of matrix Jacobi processes and its unique invariant distribution is given by the β-Jacobi ensemble:

$$\mathcal{M}^{Jac,n}_{a,b,\beta}(dx) = C^{-1}_{n,a,b,\beta} \prod_{i=1}^{n} x_i^{\frac{\beta}{2}a-1}(1-x_i)^{\frac{\beta}{2}b-1} \prod_{1\le i<j\le n} |x_j - x_i|^{\beta} dx, \tag{10}$$

for some normalization constant $C_{n,a,b,\beta}$.

We write $P^{(n)}_{d,\theta}(t)$ for the Markov semigroup associated to the solution of (8). Similarly, write $Q^{(n)}_{a,b,\theta}(t)$ for the Markov semigroup associated to the solution of (9).

We now introduce the general β analogue of the Vandermonde link (4). For $y \in W^n$ and $x \in W^{n+1}$ such that $y \prec x$, define the *Dixon-Anderson* conditional *probability* density originally introduced by Dixon at the beginning of the last century in [56] and independently rediscovered by Anderson in his study of the Selberg integral in [5] by,

$$\lambda^{\theta}_{n,n+1}(x,y) = \frac{\Gamma(\theta(n+1))}{\Gamma(\theta)^{n+1}} \prod_{1\le i<j\le n+1} (x_j - x_i)^{1-2\theta} \prod_{1\le i<j\le n} (y_j - y_i) \prod_{i=1}^{n}\prod_{j=1}^{n+1} |y_i - x_j|^{\theta-1} \mathbf{1}\left(y \prec x\right). \tag{11}$$

Denote by $\Lambda^{\theta}_{n,n+1}$, the integral operator with kernel $\lambda^{\theta}_{n,n+1}$. Observe that for $\theta = \frac{\beta}{2} = 1$ this specializes to (4). Then we have that:

Theorem 0.2. *Let $\beta \ge 1$, $d \ge 2$ and $a, b \ge 1$. Then, with $\theta = \frac{\beta}{2}$, we have the following equalities of Markov kernels, $\forall t \ge 0$,*

$$P^{(n+1)}_{d-2,\theta}(t)\Lambda^{\theta}_{n,n+1} = \Lambda^{\theta}_{n,n+1} P^{(n)}_{d,\theta}(t), \tag{12}$$

$$Q^{(n+1)}_{a-1,b-1,\theta}(t)\Lambda^{\theta}_{n,n+1} = \Lambda^{\theta}_{n,n+1} Q^{(n)}_{a,b,\theta}(t). \tag{13}$$

The semigroups $P^{(n)}_{d,\theta}(t)$ and $Q^{(n)}_{a,b,\theta}(t)$ for $\beta \neq 2$ no longer have determinantal transition kernels and a new approach is required to prove this theorem. The argument goes via first checking the desired relation at the infinitesimal level, by the action of the generators of these stochastic processes on certain symmetric functions, the Jack polynomials $J_\lambda(z;\theta)$ indexed by partitions $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge 0)$. We then conclude by a lift, using Ito's formula and the moment method, to the statement of the Theorem above. The fact that the Jack polynomials are eigenfunctions of the Calogero-Sutherland Hamiltonian (see [12]):

$$\mathcal{D}^{(n),\theta} = \sum_{i=1}^{n} z_i^2 \frac{\partial}{\partial z_i^2} + 2\theta \sum_{i=1}^{n} \sum_{1\le j\le n, j\neq i} \frac{z_i^2}{z_i - z_j}\frac{\partial}{\partial z_i}, \tag{14}$$

and also eigenfunctions of the Markov kernel Λ_θ (see [116]):

$$\left[\Lambda^{\theta}_{n,n+1} J_\lambda(\cdot;\theta)\right](x) = c(\lambda, n, \theta) J_\lambda(x;\theta)$$

is what makes the computation tractable.

Given that the algebraic framework is in place, it is natural to attempt to construct the multilevel β-Laguerre process, so that the projection on level n follows the n particle evolution (8). In particular it would be interesting to investigate whether a similar decoupling phenomenon as in [76] arises at the hard edge.

0.3 Chapter 3: Feller processes on the graph of spectra and the Hua-Pickrell measures

The problem of constructing infinite dimensional dynamics preserving certain measures coming from random matrix theory is well known. It was initiated by Spohn in his study [146] of Dyson's model (informally written) with $i \in \mathbb{Z}$ where the β_i are independent standard Brownian motions,

$$dX_i(t) = d\beta_i(t) + \sum_{j \neq i} \frac{1}{X_i(t) - X_j(t)} dt.$$

Then, these *SDEs* leave invariant the Sine_2 point process, which arises as the universal bulk scaling limit of eigenvalues of Wigner matrices. This is a determinantal point process with correlation kernel $K_{\mathsf{Sine}_2}(x, y)$ given by,

$$K_{\mathsf{Sine}_2}(x, y) = \frac{\sin(\pi (x - y))}{\pi (x - y)}.$$

In fact, originally Spohn used the theory of Dirichlet forms to construct an L^2-Markovian semigroup for the equilibrium dynamics. More recently, in several works Osada with various collaborators (see for example [124] and the references therein), combining the theory of Dirichlet forms and determinantal point processes was able to construct infinite dimensional SDEs starting from equilibrium for a wide class of such dynamics. Finally Tsai considered the equations above directly and proved well-posedness for a class of initial configurations that satisfy a certain balanced condition, see [153].

The goal of this chapter is to construct a Feller-Markov process leaving the distinguished Hua-Pickrell measures on infinite Hermitian matrices invariant. Implicit in the Feller property is the fact that we can start the process from any initial configuration. The Hua-Pickrell measures depend on a parameter $s \in \mathbb{C}$ and give rise to a determinantal point process in $\mathbb{R}^*$ with correlation kernel K_∞^{HP} having the integrable form:

$$K_\infty^{HP}(x, y) = \frac{P(x)Q(y) - Q(x)P(y)}{x - y}$$

where $P(x), Q(x)$ involve certain confluent hypergeometric functions. In the particular case of $s = 0$ and under the transformation $x \mapsto -\frac{1}{\pi x}$ it reduces to the Sine_2 process. We will elaborate on their history and remarkable properties shortly.

We take a completely different approach to attack this problem from the ones described above, which yields in some sense a stronger result. The proof takes advantage of the underlying integrable structure and goes via the construction of consistent dynamics on the "graph of spectra", making a novel use of the method of intertwiners of Borodin and Olshanski.

We will now explain this in more detail. First, we need to introduce the space Ω our dynamics takes place in. Let $\mathbb{U}(N)$ and $H(N)$ be the N-dimensional unitary group and the space of N × N Hermitian matrices. Let $\mathbb{U}(\infty)$ be the inductive limit $\varinjlim \mathbb{U}(N)$ of unitary groups. In more explicit terms an element of $\mathbb{U}(\infty)$ is an infinite matrix whose top corner is an $N \times N$ unitary matrix for some N and the rest is the identity. Let H be the space of infinite Hermitian matrices, the projective limit $\varprojlim \mathbb{H}(N)$ under the maps $\pi^N_{N-1} : H(N) \to H(N-1)$ with $\pi^N_{N-1}\left[\left(h_{ij}\right)^N_{i,j=1}\right] = \left(h_{ij}\right)^{N-1}_{i,j=1}$.

Then $\mathbb{U}(\infty)$ acts on H by conjugation: for each $u \in \mathbb{U}(\infty)$ we have a map $T_u : H \to H$ given by $T_u(h) = u^* hu$. It is a beautiful theorem of Pickrell and also Olshanski and Vershik that ergodic measures on H for this action are parametrized by the infinite dimensional space:

$$\Omega = \Big\{\omega = (\alpha^+, \alpha^-, \gamma_1, \delta) \in \mathbb{R}^{2\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R} |$$

$$\alpha^+ = (\alpha^+_1 \geq \alpha^+_2 \geq \cdots \geq 0)\ ;\ \alpha^- = (\alpha^-_1 \geq \alpha^-_2 \geq \cdots \geq 0);$$

$$\gamma_1 \in \mathbb{R}\ ;\ \delta \geq 0; \sum (\alpha^+_i)^2 + \sum (\alpha^-_i)^2 \leq \delta \Big\},$$

$$\gamma_2 = \delta - \sum (\alpha^+_i)^2 + \sum (\alpha^-_i)^2.$$

An ergodic measure M_ω is then determined by its Fourier transform as follows:

$$\int_{X \in H} e^{iTr(diag(r_1,\cdots,r_n,0,0,\cdots)X)} M_\omega(dX) = \prod_{j=1}^{n} F_\omega(r_j),$$

$$F_\omega(x) = e^{i\gamma_1 x - \frac{\gamma_2}{2}x^2} \prod_{k=1}^{\infty} \frac{e^{-i\alpha^+_k x}}{1 - i\alpha^+_k x} \prod_{k=1}^{\infty} \frac{e^{i\alpha^-_k x}}{1 + i\alpha^-_k x}\ .$$

The parameters in Ω have certain meaning, which was first explained in Olshanski's and Vershik's proof. Let $h \in H$ be a random matrix with law M_ω. Consider the spectrum of its top $N \times N$ corner:

$$\text{spectrum}\left[(h_{ij})^N_{i,j=1}\right] = \left(x^{(N)}_1 \leq \cdots \leq x^{(N)}_N\right).$$

Embed the eigenvalues in Ω as follows:

$$
\begin{aligned}
\alpha_i^+\left(x^{(N)}\right) &= \begin{cases} \frac{\max\{x^{(N)}_{N+1-i},0\}}{N} & i = 1,\cdots,N \\ 0 & i = 1 = N+1, N+2,\cdots \end{cases}, \\
\alpha_i^-\left(x^{(N)}\right) &= \begin{cases} \frac{\max\{-x^{(N)}_{i},0\}}{N} & i = 1,\cdots,N \\ 0 & i = 1 = N+1, N+2,\cdots \end{cases}, \\
\gamma_1\left(x^{(N)}\right) &= \sum_{i=1}^{\infty} \alpha_i^+\left(x^{(N)}\right) - \sum_{i=1}^{\infty} \alpha_i^-\left(x^{(N)}\right), \\
\delta\left(x^{(N)}\right) &= \sum_{i=1}^{\infty} \left(\alpha_i^+\left(x^{(N)}\right)\right)^2 + \sum_{i=1}^{\infty} \left(\alpha_i^-\left(x^{(N)}\right)\right)^2 .
\end{aligned}
$$

We denote this embedding by $\mathfrak{r}_N$. Then, Olshanski and Vershik [123] have shown that (see also Borodin and Olshanski [25]):

$$
\begin{aligned}
\alpha_i^+\left(x^{(N)}\right) &\xrightarrow{d} \alpha_i^+ \ , \forall i \geq 1, \\
\alpha_i^-\left(x^{(N)}\right) &\xrightarrow{d} \alpha_i^- \ , \forall i \geq 1, \\
\gamma_1\left(x^{(N)}\right) &\xrightarrow{d} \gamma_1, \\
\delta\left(x^{(N)}\right) &\xrightarrow{d} \delta.
\end{aligned}
$$

We now introduce the Hua-Pickrell measures on $H(N)$, for $s \in \mathbb{C}$ with $\Re(s) > -\frac{1}{2}$ so that they are finite:

$$
\mathsf{M}_{HP}^{s,N}(dX) = const \times \det\left((I+iX)^{-s-N}\right)\det\left((I-iX)^{-\bar{s}-N}\right) \times dX, \tag{15}
$$

where dX denotes Lebesgue measure on $H(N)$. These distinguished measures were first studied for a real parameter s by Hua Luogeng in the 50's in his book [81] on harmonic analysis in several complex variables and were later in the 80's rediscovered independently by Pickrell [128] in the context of Grassmann manifolds. Then, around the turn of the millennium, Neretin studied a generalization allowing for a complex parameter s as part of a larger program in [108]. It is a remarkable fact that these measures are consistent with respect to cutting corners:

$$
\left[\left(\pi_{N-1}^N\right)_* \mathsf{M}_{HP}^{s,N}\right](dX) = \mathsf{M}_{HP}^{s,N-1}(dX).
$$

Thus, by Kolmogorov's theorem, they give rise to a measure M_{HP}^s on H which turns out to be non-ergodic. There exists a measure μ_{HP}^s on Ω which then gives the ergodic decomposition as follows:

$$\mathsf{M}^s_{HP}(dX) = \int_\Omega M_\omega(dX)\mu^s_{HP}(d\omega).$$

Under the so called forgetting map which disregards γ_1, γ_2 and any $\alpha_i^{\pm}$'s which are zero μ^s_{HP} gives rise to the determinantal point process with kernel K^{HP}_∞. This problem of ergodic decomposition was first investigated by Borodin and Olshanski in [25]. More recently it was intensely studied by Bufetov and Qiu, the complete solution clarifying the roles of γ_1 and γ_2 given in [130].

It is for the measure μ^s_{HP} that we construct stochastic dynamics. The argument proceeds as follows: One considers the 'graph' of spectra (a name originating with Kerov), which is not really a graph in the strict sense but a projective system of measures. It consists of levels with 'vertices' $W^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 \leq \cdots \leq x_n\}$ and the links or Markov kernels (for x in the interior of W^n):

$$\Lambda^n_{n-1}(x, dy) = \frac{(n-1)!\Delta_{n-1}(y)}{\Delta_n(x)} \mathbf{1}\,(y \prec x)\, dy_1 \cdots dy_{n-1}.$$

In fact, this Markov kernel arises as the conditional distribution of the eigenvalues of the $(n-1) \times (n-1)$ corner of a random $n \times n$ unitarily invariant Hermitian matrix with eigenvalues $x_1 \leq \cdots \leq x_n$. More formally, with G being a bounded class (depending only on the eigenvalues) function on $H(n-1)$ and the expectation $\mathbb{E}$ taken with respect to Haar measure on $\mathbb{U}(n)$:

$$\left[\Lambda^n_{n-1} G\right](x_1, \cdots, x_n) = \mathbb{E}_{u \in \mathbb{U}(n)}\left[G\left[\pi^n_{n-1}\left(u^* \mathrm{diag}\left[x_1, \cdots, x_n\right] u\right)\right]\right].$$

This is a result of Baryshnikov [14], but in fact these computations go back to Gelfand and Naimark's book on representation theory [74]. Observe that this is the same Markov kernel (4), the Vandermonde link, arising in Warren's construction for reflecting interlacing Brownian motions described in Chapter 1. Moreover, note that this representation makes sense even for x on the boundary of W^n.

Now, given such a system of links we will say that a sequence of measures $\{\mu_N\}_{N\geq 1}$ is coherent if:

$$\mu_{N+1}\Lambda^{N+1}_N = \mu_N.$$

These coherent sequences form a convex set and the extremal ones form the boundary of this system of links. It is a brief observation of Borodin and Olshanski in [25] that Ω can be identified with the boundary of the 'graph' of spectra. In Chapter 3 I expand upon this observation, providing explicit expressions for Markov kernels $\Lambda^\infty_N : \Omega \to W^N$ and (simple but non-trivial) proofs of the Feller property for all the Markov kernels involved.

To restate this result in more classical Markov process language, see section 4.1.3

of [168]: the 'graph' of spectra is a Markov chain (living in varying state spaces) moving backwards from time ∞ to time 1 and Ω forms its entrance boundary.

Then using a general formalism introduced by Borodin and Olshanski [28] in order to construct a Feller process on Ω having μ^s_{HP} as its unique invariant measure it suffices to construct *consistent* (Feller) processes with evolution semigroups $P^{s,N}_{HP}(t)$ on each level set W^N with $\mu^{s,N}_{HP}$ being their unique invariant measures. Here by consistent we mean that the semigroups $P^{s,N}_{HP}(t)$ are intertwined with respect to the links Λ^{N+1}_N.

We construct the needed semigroup $P^{s,N}_{HP}(t)$ as the transition operator associated to the solution $X^{s,N}_{HP}(t) = (X_1(t), \cdots, X_N(t))$ of the following system of repulsive SDEs:

$$dX_i(t) = \sqrt{2(X_i^2(t)+1)}dW_i(t) + \left[(2-2N-2\Re(s))\,X_i(t) + 2\Im(s) + \sum_{j\neq i}\frac{2(X_i^2(t)+1)}{X_i(t)-X_j(t)}\right]dt. \quad (16)$$

Moreover, its kernel is given as a Doob's transformed Karlin-McGregor determinant. Then, using the results from Chapter 1 and some manipulations it is possible to show:

Proposition 0.3.

$$P^{s,N+1}_{HP}(t)\Lambda^{N+1}_N = \Lambda^{N+1}_N P^{s,N}_{HP}(t)\ ,\ \forall t \geq 0,\ \forall N \geq 1.$$

Moreover, for $\Re(s) > -\frac{1}{2}$ and $N \geq 1$ the measure $\mu^{s,N}_{HP}$ is the unique invariant measure of $P^{s,N}_{HP}(t)$.

From which we conclude:

Theorem 0.4. *There exists a unique Feller semigroup $P^{s,\infty}_{HP}(t)$ on Ω that is consistent with the semigroups $\{P_N(t)\}_{N\geq 1}$, so that for $f \in C_0\left(W^N\right)$,*

$$P^{s,\infty}_{HP}(t)\Lambda^{\infty}_N f = \Lambda^{\infty}_N P^{s,N}_{HP}(t) f\ ,\ \forall t \geq 0,\ \forall N \geq 1.$$

Moreover, if $\Re(s) > -\frac{1}{2}$ the measure μ^s_{HP} is its unique invariant measure.

Finally, we have the following concrete approximation of infinite process $\left(\mathsf{X}^{s,\infty}_{HP}(t); t \geq 0\right)$ associated to $P^{s,\infty}_{HP}(t)$ by the finite dimensional processes $\left(X^{s,N}_{HP}(t); t \geq 0\right)$:

Proposition 0.5. *Let μ be any probability measure on Ω. Let μ_N be the coherent measures on W^N corresponding to μ. Then, for any $F \in C_0(\Omega)$ and fixed $t \geq 0$:*

$$\mathbb{E}_{\mu}\left[F\left(\mathsf{X}^{s,\infty}_{HP}(t)\right)\right] = \lim_{N\to\infty}\mathbb{E}_{\mu_N}\left[F\left(\mathrm{r}_N\left(X^{s,N}_{HP}(t)\right)\right)\right].$$

0.4 Chapter 4: Matrix Bougerol identity

This chapter is concerned with extensions to the matrix setting of identities involving exponential functionals of Brownian motions. It originates from my work in Chapter 3 on dynamics for the Hua-Pickrell measures; I provide a matrix extension of a very well known identity due to Bougerol involving one-dimensional Brownian motions obtaining a construction of the matrix Hua-Pickrell measures in display (15) as stochastic integrals along the way. A key role is played by a matrix valued process whose eigenvalues evolve according to the diffusion given in (16).

To describe this, let $(\beta_t; t \geq 0)$ and $(\gamma_t; t \geq 0)$ be two independent standard Brownian motions starting from 0. Then, for *fixed* $t \geq 0$, we have the following equalities in law,

$$\sinh(\beta_t) \overset{\text{law}}{=} \int_0^t e^{\beta_s} d\gamma_s \overset{\text{law}}{=} \gamma\left(\int_0^t e^{2\beta_s} ds\right). \tag{17}$$

Bougerol proved this identity in his study of convolution powers of probabilities on certain solvable groups. Later Marc Yor and coauthors in [13] and [4] gave a simple diffusion theoretic proof of this identity obtaining along the way results such as the following: if we denote by $\left(\beta_t^{(-\nu)}; t \geq 0\right)$ and $\left(\gamma_t^{(-\mu)}; t \geq 0\right)$ two independent standard Brownian motions with drifts $-\nu$ and $-\mu$ respectively the law of the functional for $\nu > 0$

$$\int_0^\infty e^{\beta_t^{(-\nu)}} d\gamma_t^{(-\mu)} \tag{18}$$

has density, with respect to Lebesgue measure, given by,

$$f_{\nu,\mu}(x) = \mathrm{c}_{\nu,\mu} \frac{e^{-2\mu \arctan(x)}}{(1+x^2)^{\nu+\frac{1}{2}}}.$$

There is a closely related and equally well-known identity in one dimension, originally proven by Dufresne in [61]: Consider the functional,

$$a_t^{(-\nu)} = \int_0^t e^{2\beta_s^{(-\nu)}} ds.$$

Then, for $\nu > 0$,

$$a_\infty^{(-\nu)} \overset{\text{law}}{=} \frac{1}{2\xi_\nu} \tag{19}$$

where ξ_ν is a Gamma distributed random variable with density $\frac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x}$.

The study of these one-dimensional identities, in the general setting of exponential functionals of Brownian motions, has in fact led to many developments in integrable probability, although the initial motivation were applications in mathematical finance and insurance. For example, Matsumoto and Yor derived the geometric Levy and Pitman theorems which are intimately related to the exact solvability of the Brownian random polymer (or so called O'Connell-Yor polymer) and its connection to the Quantum Toda lattice [112].

Recently, Rider and Valko in [135] have proven a matrix version of Dufresne's identity, obtaining in place of an inverse Gamma random variable, the inverse Wishart laws. A further motivation for the investigation of this chapter was to answer Rider and Valko's question whether other well known matrix laws can be constructed by this diffusion theoretic approach, or "Dufresne procedure" as referred to in [135].

Finally we mention that, Marc Yor had an ongoing program for some time, trying to obtain higher dimensional generalizations of Bougerol's identity and study their ramifications ([38]). In the last few years, some interesting progress was made in his joint work with Bertoin and Dufresne ([16]), where a generalization involving a (still) one-dimensional process and its local time was discovered. However, the contribution of this chapter provides the first truly multi-dimensional extension.

We now explain our result. $(\boldsymbol{W}_t; t \geq 0)$ will denote an $N \times N$ matrix whose entries consist of independent (scalar) complex Brownian motions. We will denote by $\left(\boldsymbol{M}_t^{(\nu)}; t \geq 0\right)$ the matrix analogue of the exponential of complex Brownian motion with drift ν, given by the solution to the following linear matrix Stochastic Differential Equation (SDE), starting from $\boldsymbol{M}_0^{(\nu)} = \boldsymbol{I}$,

$$d\boldsymbol{M}_t^{(\nu)} = \frac{1}{\sqrt{2}}\boldsymbol{M}_t^{(\nu)} d\boldsymbol{W}_t + \nu \boldsymbol{M}_t^{(\nu)} dt.$$

This is essentially a Brownian motion on the group $\mathbb{GL}_N(\mathbb{C})$. Moreover, with $s \in \mathbb{C}$ a complex parameter consider the following matrix SDE taking values in $\mathbf{H}(N)$ (if $\boldsymbol{X}_0 \in \mathbf{H}(N)$), where $(\boldsymbol{\Gamma}_t; t \geq 0)$ denotes a complex Brownian matrix,

$$d\boldsymbol{X}_t = d\boldsymbol{\Gamma}_t \sqrt{\frac{\boldsymbol{I} + \boldsymbol{X}_t^2}{2}} + \sqrt{\frac{\boldsymbol{I} + \boldsymbol{X}_t^2}{2}} d\boldsymbol{\Gamma}_t^\dagger + \left[(-N - 2\Re(s))\boldsymbol{X}_t + 2\Im(s)\boldsymbol{I} + \mathrm{Tr}\left(\boldsymbol{X}_t\right)\boldsymbol{I}\right] dt. \tag{20}$$

This is the matrix process that has (16) as its eigenvalue evolution. Moreover, this is a Hermitian analogue of (a general version of) $\sinh(\beta_t)$. To see the analogy more clearly, note that,

$$d\sinh(\beta_t) = \left(1 + \sinh^2(\beta_t)\right)^{\frac{1}{2}} d\beta_t + \frac{1}{2}\sinh(\beta_t) dt.$$

Thus, (modulo normalization constants) to arrive at (20) we simply replaced the scalar (quadratic, with no real roots) diffusion and (linear) drift coefficients by their (symmetrized) matrix analogues. Finally we will write throughout $\left(\boldsymbol{B}_t^{(\mu)}; t \geq 0\right)$ for a drifting complex Brownian matrix with drift $\mu \in \mathbb{R}$, given by,

$$\boldsymbol{B}_t^{(\mu)} = \boldsymbol{B}_t + \mu \boldsymbol{I} t$$

for a complex Brownian matrix $(\boldsymbol{B}_t; t \geq 0)$ which is *independent* of $(\boldsymbol{W}_t; t \geq 0)$.

First, the matrix analogue of the exponential functional distribution (18), a construction of the Hua-Pickrell measures $\mathsf{M}_{HP}^{s,N}$ as stochastic integrals:

Theorem 0.6. *Let $\Re(s) > -\frac{1}{2}$. With $\nu = \Re(s) + \frac{N}{2}$, $\mu = \sqrt{2}\Im(s)$, then,*

$$\int_0^\infty \boldsymbol{M}_t^{(-\nu)} \left(\frac{d\boldsymbol{B}_t^{(\mu)} + d\left(\boldsymbol{B}_t^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_t^{(-\nu)}\right)^\dagger \tag{21}$$

is distributed as $\mathsf{M}_{HP}^{s,N}$.

In order to prove this we first need the following Hermitian version of Bougerol's identity (17).

Theorem 0.7. *With $\nu = \Re(s) + \frac{N}{2}$, $\mu = \sqrt{2}\Im(s)$, denote by $\tilde{\boldsymbol{X}}_t^{\mu,\nu}$ the unique solution of (20) starting from the $\boldsymbol{0}$ matrix. Then, for fixed $t > 0$,*

$$\tilde{\boldsymbol{X}}_t^{\mu,\nu} \overset{law}{=} \int_0^t \boldsymbol{M}_u^{(-\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(-\nu)}\right)^\dagger. \tag{22}$$

We are able to prove Theorem 0.7 because remarkably the matrix SDE (20) can be solved explicitly. This is due to the fact that (20) has an alternative linear matrix SDE representation. Then, we make use of a time reversal argument to obtain the stated result. In order to prove Theorem 0.6, along with Theorem 0.7, we use another non-trivial result, namely that the evolution (20) leaves $\mathsf{M}_{HP}^{s,N}$ invariant (this was the reason for its introduction in the first place). Finally, some estimates are required to show that the integrals indeed converge for $\Re(s) > -\frac{1}{2}$.

0.5 Chapter 5: Random surface growth and Karlin-McGregor polynomials

Dynamics on branching graphs The first part of this chapter is concerned with the study of dynamics on branching graphs. It forms in some sense the discrete analogue of the first chapter.

The study of branching graphs considered here originates with questions of asymptotic representation theory. The story begins with the problem of classification of the extreme or indecomposable characters of the infinite symmetric $S(\infty) = \varinjlim S(n)$ and infinite-dimensional unitary groups $\mathbb{U}(\infty) = \varinjlim \mathbb{U}(n)$, the inductive limits of the corresponding chains of subgroups. The extreme characters of $S(\infty)$ were first considered by Thoma in the 60's [150] and then a decade later the characters of $\mathbb{U}(\infty)$ by Voiculescu [161]. It was then realized that the classification of extreme characters of both groups were implicitly contained in earlier works obtained in the 50's by Schoenberg and his school, see [1], [2] on total positivity. These relied on some deep function theoretic results to classify totally positive Toeplitz matrices.

Then, in the 80's came the pioneering work of Vershik and Kerov [160],[159] who revolutionized the field by introducing the probabilistic approach of approximating the extreme characters by their finite dimensional analogues or equivalently by calculating the boundary of the associated branching graph using the ergodic method of Vershik [158]. The associated branching graph here refers to a graded graph whose vertices are given by the parametrization of irreducible characters of the finite N subgroups (e.g. $S(N)$ or $\mathbb{U}(N)$) and its edges encode the branching (or decomposition) of irreducibles when restricted to a smaller subgroup. The graph associated to $S(\infty)$ is the Young graph and to $\mathbb{U}(\infty)$ is the Gelfand-Tsetlin graph. Their vertex sets consist of Young diagrams (equivalently partitions) and signatures (weakly ordered integer sequences) respectively and the weight on their edges can also be seen as the Pieri and branching coefficients of the Schur polynomials. This is somehow the beginning of how the theory of symmetric functions enters the picture and we shall say more in the appendix of this chapter. It would be impossible to survey the massive literature here, the reader is referred to the beautiful thesis of Kerov [99] where many deep ideas which permeate the field can originally be found and the very recent book of Borodin and Olshanski [31] for a more up to date and friendly exposition.

To formalize some ideas we briefly describe the general framework. This is essentially combinatorial and probabilistic and makes no reference to representation theory. We will be concerned with graded graphs Γ, with vertex sets $\sqcup_N V_N$ such that each V_N is countable. For each $x \in V_{N+1}$ there is at least one edge but not infinitely many connecting it to a vertex in V_N and for each $y \in V_N$ there is at least one edge connecting it to a vertex in V_{N+1}. We assign certain multiplicities or weights to each edge denoted by $\mathsf{mult}(x,y)$ and define the weight of all paths ending at $x \in V_N$, for $N \geq 2$ recursively by,

$$\mathsf{dim}_N(x) = \sum_{y \in V_{N-1}} \mathsf{mult}(x,y)\mathsf{dim}_{N-1}(y).$$

Note that, we need to specify an initial weight/dimension $\mathsf{dim}_1(\cdot)$ for the vertices in the set V_1. Then one can define a Markov kernel, or cotransition probabilities, from V_{N+1} to V_N as follows,

$$\Lambda_N^{N+1}(x,y) = \frac{\mathsf{mult}(x,y)\mathsf{dim}_N(y)}{\mathsf{dim}_{N+1}(x)}.$$

These can be thought of as determining a Markov chain evolving backwards in discrete time N and moving down the levels of the graph (as in the 'graph' of spectra of Chapter 3). The boundary of the graph is then given by the set of extremal coherent probability measures; namely sequences of (probability) measures $\{\mu_N\}_{N\geq 1}$ on $\{V_N\}_{N\geq 1}$ that satisfy,

$$\mu_{N+1}\Lambda_N^{N+1} = \mu_N$$

and that cannot be decomposed into convex combinations of other such sequences. By a general abstract theorem, see [168], the boundary is isomorphic to a Borel space Ω_Γ that comes equipped with Markov kernels $\Lambda_N^\infty : \Omega_\Gamma \to V_N$. It is a remarkable fact, and usually very hard to prove, that in many interesting situations it is possible to describe explicitly the space Ω_Γ and also the Markov kernels Λ_N^∞.

We now describe how we can associate a branching graph $\Gamma = \Gamma(\{G(N)\})$ to a chain of finite or compact groups:

$$G(1) \subset G(2) \subset \cdots G(N-1) \subset G(N) \subset \cdots .$$

The vertices at level N are the equivalence classes of irreducible representations of the group $G(N)$. Now, pick a representation π_v corresponding to each vertex v. Then, two vertices u and v at levels $N-1$ and N respectively are joined by an edge with $\mathsf{mult}(u,v) = m$ if π_u enters m times into the decomposition of the restriction $\pi_v \downarrow G(N-1)$.

The complete description of the boundary is just the beginning of many possible directions one could take and of questions to be tackled. For example, the decomposition of a measure on the boundary Ω_Γ corresponding to certain distinguished coherent sequences of measures; the so-called zw-measures on the Gelfand-Tsetlin graph or the z-measures on the Young graph are the problems of harmonic analysis on the infinite dimensional unitary and infinite symmetric groups respectively, see for example [120] and [26]. From the standpoint of noncommutative harmonic analysis this task is the analogue of the question of decomposition of the regular representation and its relatives into irreducibles.

Another direction is the construction of stochastic dynamics on the graph. As briefly explained in the paragraph corresponding to Chapter 3, in the past decade, Borodin and Olshanski introduced in [28] the method of intertwiners for constructing Markov processes on the boundaries of branching graphs (and more generally projective chains). The key input to this approach are intertwining relations,

$$P_N(t)\Lambda_{N-1}^N = \Lambda_{N-1}^N P_{N-1}(t), \ t \geq 0, N \geq 1$$

for a sequence $(P_N(t); t \geq 0)$ of transition semigroups on individual levels V_N of the graph. They then made use of this theory to construct a Markov process leaving invariant the zw-measures on the boundary of the Gelfand-Tsetlin graph.

More recently, Cuenca in [50] performed the same task for the type-BC graph which has a representation theoretic origin as well. For certain values of its multiplicies it describes the branching of the irreducible characters of the Lie groups $\{SO(2N+1)\}_{N\geq 1}$, $\{Sp(2N)\}_{N\geq 1}$ and $\{O(2N)\}_{N\geq 1}$. Vertices at level N are now given by *positive* signatures of length N and two vertices are connected if they satisfy a certain kind of type-BC interlacing, see section 5.4 for more details.

In subsection 5.5 it is shown how both of these results follow as corollaries from a 'master' intertwining relation and now proceed to explain this further. As already anticipated from Chapter 1 a key role is played by Siegmund duality between birth and death chains $X(t)$, living in $I = \mathbb{N}$ or bilateral ones taking values in $I = \mathbb{Z}$. If the chain $X(t)$ is at site x it jumps to the right with rate $\lambda(x)$ and to the left with rate $\mu(x)$. We will assume that

$\lambda(x), \mu(x) > 0$, for all $x \in \mathbb{Z}$ in the bilateral case and $\mu(0) = 0$ in case of $I = \mathbb{N}$, i.e. that 0 is reflecting. Let $p_t(i, j) = \mathbb{P}_i\left(X(t) = j\right)$ be its transition density and $\pi(\cdot)$ the measure with respect to which it is reversible. We now define the Siegmund dual chain $\hat{X}(t)$ with rates to the right $\hat{\lambda}(x) = \mu(x+1)$ and and to the left $\hat{\mu}(x) = \lambda(x)$. Observe that in case $I = \mathbb{N}$ since $\hat{\mu}(0) > 0$ the chain is absorbed at -1. As in the continuum setting Siegmund duality means that if we consider two copies $X(t)$ and $\hat{X}(t)$ then for $x, y \in I$ and $t \geq 0$ we have:

$$\mathbb{P}_x\left(X(t) \leq y\right) = \mathbb{P}_y\left(\hat{X}(t) \geq x\right).$$

We will now consider ordered configurations of particles on a lattice with no two per site, that abusing notations we will denote as in the continuum setting by $W^N = \{(x_1, \cdots, x_N) : x_1 < x_2 < \cdots < x_N\}$. Both partitions and signatures after an invertible shift can be transformed to such configurations. We will say that $y \in W^N$ interlaces with $x \in W^{N+1}$ and denote this by $y \prec x$ if: $x_1 \leq y_1 < x_2 \leq \cdots < x_{n+1}$. We will also say that $y \in W^N$ interlaces with $x \in W^N$ if: $y_1 \leq x_1 < y_2 \leq \cdots \leq x_n$ and also denote this by $y \prec x$. Note that, there is a minor asymmetry in the locations of inequalities and strict inequalities.

We shall denote the Karlin-McGregor semigroup associated to n $\mathcal{D}$-chains killed when they intersect by $\left(P_t^n; t \geq 0\right)$. This, analogously to the continuum, has transition kernel given by:

$$p_t^N(x, y) = \det(p_t(x_i, y_j))_{i,j=1}^N.$$

Similarly, we will write $\left(\hat{P}_t^N; t \geq 0\right)$ for the one associated to $\hat{\mathcal{D}}$-chains. We also define the positive kernels:

$$\left(\Lambda_{n,n+1} f\right)(x) = \sum_{y \prec x} \prod_{i=1}^{n} \hat{\pi}(y_i) f(y),$$

$$\left(\Lambda_{n,n} f\right)(x) = \sum_{y \prec x} \prod_{i=1}^{n} \pi(y_i) f(y).$$

Then we have, for $t \geq 0$:

$$P_t^{N+1} \Lambda_{N,N+1} = \Lambda_{N,N+1} \hat{P}_t^N, \tag{23}$$

$$\hat{P}_t^N \Lambda_{N,N} = \Lambda_{N,N} P_t^N. \tag{24}$$

By performing suitable Doob's h-transformations to these relations, the results of Borodin-Olshanski and Cuenca then follow, see subsection 5.5.

Random surface growth and determinantal point processes The second part of this chapter is concerned with the stochastic growth of discretized surfaces. Such random height functions arise in many models in statistical physics, such as domino tilings, harmonic crystals, Ginzburg-Landau $\nabla\phi$ interface models, see e.g. [73],[98] and for a deep study of translation invariant random surfaces the reader is referred to the PhD thesis of Sheffield [143].

Here, we will focus our attention to special models with a lot of underlying structure. In particular, we will be concerned with the exact computation of the correlation kernel of a determinantal process associated to a random growth model with a wall. The remarkable fact about this process is that it is possible to introduce arbitrary inhomogeneities in a one dimensional section of the surface and still retain the exact solvability.

First, we describe a related and a bit simpler model introduced by Borodin and Ferrari [21], that initiated the study of this sort of models. The dynamics take place in a (discrete) Gelfand-Tsetlin pattern, namely level N consists of configurations in W^N and consecutive levels interlace. It is possible to associate to this configuration a height function and so the dynamics can be viewed as a stochastic growth model. Equivalently such a pattern is a path in the Gelfand-Tseltin graph.

Each particle has an exponential clock of rate 1 for jumping to the right. The particles interact through the so-called push-block dynamics: There's a hierarchy for the particles, lower level ones can be thought of as heavier or more important. If the clock of the particle X_k^n rings first, it attempts to jump to the right by one unit. It first looks at the $(n-1)^{th}$ level to check whether it is blocked, namely if $X_k^{n-1} = X_k^n$. In case it is, nothing happens, otherwise it moves by one to the right, possibly triggering some pushing moves. Namely if the interlacing is no longer preserved with the particle labelled X_{k+1}^{n+1} then X_{k+1}^{n+1} also moves (instantaneously) to the right by one. This pushing is propagated to higher levels.

Another reason for studying $(2+1)$-dimensional models is that $(1+1)$-dimensional interface growth models actually arise as one dimensional sections of the surface. For example, as can be easily seen, in the model just described the evolution of the left-most particles $(X_1^1(t), \cdots, X_1^N(t); t \geq 0)$ is that of TASEP and the evolution of the right-most ones $(X_1^1(t), \cdots, X_N^N(t); t \geq 0)$ is that of PushASEP. This, has already been observed in the continuum in chapter 1, with diffusive particles with one-sided collisions. An interesting observation is that in the continuum the interactions at the right and left edges of the pattern are much more similar.

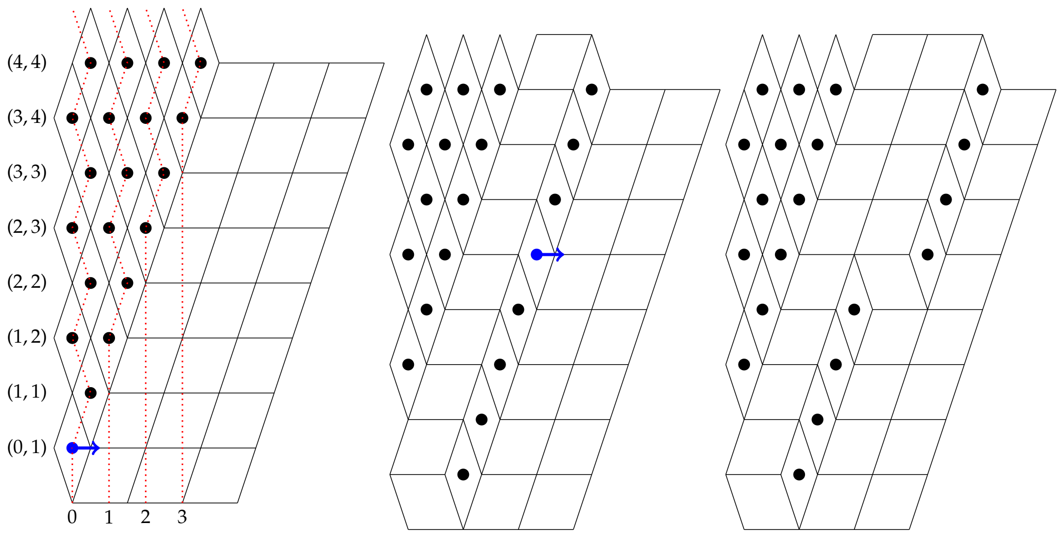
Borodin and Ferrari were able to show that along so called space-like paths this point process is determinantal and compute explicitly the correlation kernel when started from the fully packed initial condition. Then, many asymptotic properties of this growth model were studied, in particular proving that it belongs to the anisotropic KPZ universality class (see [169]), i.e. having $\sqrt{log(t)}$ fluctuations. More recently, generalizations of this model where studied by Toninelli [151] using non-integrable techniques, however as expected when some structure is lost the results obtained are not as detailed, see also Ferrari and Chhita [48].

Around the same time Borodin and Kuan studied a model in [23] with a wall, related to the representation theory of $\mathbb{O}(\infty)$, the infinite-dimensional orthogonal group. The associated particle dynamics again take place in an interlacing array, that is called a symplectic Gelfand-Tsetlin pattern:

$$\mathbb{GT}_{\mathbf{s}}(\infty) = \left\{ \mathbb{X} = \left(\mathbb{X}^{(0,1)}, \mathbb{X}^{(1,1)}, \mathbb{X}^{(1,2)}, \cdots \right) : \mathbb{X}^{(i-1,i)}, \mathbb{X}^{(i,i)} \in W^i, \mathbb{X}^{(i-1,i)} \prec \mathbb{X}^{(i,i)} \prec \mathbb{X}^{(i,i+1)} \right\}.$$

As is hopefully clear from the figure the correspondence between the particle configuration and the height function $h_t(x_1, x_2)$ defined over the horizontal plane in the picture below is given as follows:

$$h_t(x_1, x_2) = \sharp\{\text{particles to the right of horizontal level } x_1 \text{ at vertical level } x_2 \text{ at time } t\}.$$



The particles again evolve through the push-block dynamics. Each particle now has two independent exponential clocks of rate 1 for jumping to the right and to the left. When the clock of a particle rings, say the one corresponding to jumps to the right, it checks to see if it is blocked, i.e. if the move to the right will break the interlacing with the *preceding* level. If it is not blocked it moves to the right by one. This possibly triggers a pushing move so that interlacing with the *next* level is maintained. This pushing is propagated to levels higher up. See figure above for an illustration.

More recently, Cerenzia and Kuan [47] (see also Cerenzia [46]) studied a generalization of this, that they called Jacobi growth process, where the rates for jumping to the right on odd levels (similarly the rates to the left) depend on the following rational way on the horizontal position x:

$$\frac{x + a_1(\alpha, \beta)}{x + a_2(\alpha, \beta)} \frac{x + a_3(\alpha, \beta)}{x + a_4(\alpha, \beta)}$$

where the $a_i(\alpha, \beta)$ are certain explicit constants depending on the parameters α, β of the Jacobi weight. Rates on even levels also have a similar rational expression form.

The key insight was that these alternating rates on odd and even levels are given by the Siegmund dual rates. This led me to introduce the following generalization of the models above. The dynamics are as before but each particle on odd levels has independent exponential clocks of jumping to the right and to the left given by two (essentially) arbitrary (strictly) positive functions $\lambda(x)$ and $\mu(x)$. Particles on even levels move according to the dual rates $\hat{\lambda}(x) = \mu(x+1)$ and $\hat{\mu}(x) = \lambda(x)$. We also let Ξ^t denote the distribution of this point process at time t.

Using the intertwining relations in the previous subsection and general results for constructing multilevel processes interacting through push-block dynamics it can be seen that evolved Gibbs measures are given as products of determinants. Then making use (of a variant) of the famous Eynard-Mehta Theorem (see [35]) it is then standard that there is an underlying determinantal structure for this point process. However, to compute its correlation kernel $\mathcal{K}^t$ explicitly one needs to either invert a Gram matrix or solve a biorthogonalization problem, which is usually a formidable task.

It is at this point that a second insight is required. We make use of the spectral theory for one-dimensional birth and death chains first developed by Karlin and McGregor in [92], [93]. More precisely we define the polynomials $Q_i(x)$ through the three term recurrence:

$$Q_0(x) = 1, -xQ_0(x) = -(\lambda(0) + \mu(0))Q_0(x) + \lambda(0)Q_1(x),$$
$$-xQ_n(x) = \mu(n)Q_{n-1}(x) - (\lambda(n) + \mu(n))Q_n(x) + \lambda(n)Q_{n+1}(x).$$

These are orthogonal with respect to the *spectral measure* $\mathrm{d}\mathfrak{w}(x)$ on $\mathbb{R}_+$ with support $\mathfrak{I}$,

$$\int_0^\infty Q_i(x)Q_j(x)\mathrm{d}\mathfrak{w}(x) = \frac{1}{\pi(j)}\delta_{ij}.$$

If we view $\mathcal{D}_n$, the generator of the birth and death chain with rates $(\lambda(\cdot), \mu(\cdot))$, as a difference operator in the discrete variable n, then the three term recurrence is actually an eigenfunction relation:

$$\mathcal{D}_n Q_n(x) = -xQ_n(x).$$

These ingredients provide the following spectral expansion for the transition density of the chain:

$$p_t(i, j) = \pi(j)\int_0^\infty e^{-tx}Q_i(x)Q_j(x)\mathrm{d}\mathfrak{w}(x).$$

One can also define the polynomials $\hat{Q}_k$ and measure $\hat{\mathfrak{w}}$ associated to the Siegmund dual chain and many relations exist between these dual polynomials, which can be found in Section 5.6.

Only after expressing, the entries of the determinants appearing in the distribution of the growth process starting from the fully packed initial condition in terms of these one dimensional orthogonal polynomials and the spectral measures, that it was possible to see what the solution to the biorthogonalization problem is. Then one verifies that it is indeed the solution. Finally after some more algebraic manipulations we arrive at:

Theorem 0.8. *Let $\mathfrak{I}$ be compact then the correlation functions $\{\rho_k^t\}_{k\geq 0}$ of Ξ^t are determinantal:*

$$\rho_k^t(z_1,\cdots,z_k) \overset{\text{def}}{=} \Xi^t(\{E \in \mathbb{GT}_{\mathbf{s}}(\infty) \text{ s.t. } \{z_1,\cdots,z_k\} \subset E\}) = \det\left(\mathcal{K}^t(z_i,z_j)\right)_{i,j=1}^k \tag{25}$$

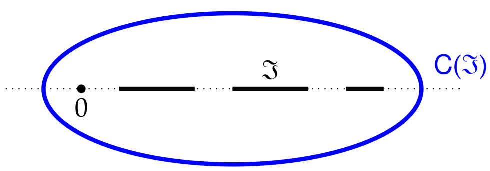
where $\mathcal{K}^t$ is given by,

$$\begin{aligned}\mathcal{K}^t\left(((n_1,n_2),i),(m_1,m_2),j))\right) = \frac{1}{2\pi \mathrm{i}}\oint_{u\in \mathsf{C}(\mathfrak{I})}\int_{x\in\mathfrak{I}} \tilde{\mathcal{P}}_j(u)\bar{\mathcal{P}}_i(x)\frac{x^{n_2}}{u^{m_2}}\frac{e^{-tx}}{(x-u)e^{-tu}}d\mathfrak{m}(x)du \\ +\mathbf{1}\left((n_1,n_2)\geq(m_1,m_2)\right)\int_{\mathfrak{I}}\bar{\mathcal{P}}_i(x)x^{n_2-m_2}\tilde{\mathcal{P}}_j(x)d\mathfrak{m}(x)\end{aligned} \tag{26}$$

and,

$$(\bar{\mathcal{P}},\tilde{\mathcal{P}},\mathfrak{m}) = \begin{cases}(\pi_i Q_i, Q_j, \mathfrak{w}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n+1),(m,m+1)\\ (\pi_i Q_i, \hat{Q}_j, \mathfrak{w}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n+1),(m,m)\\ (\hat{\pi}_i \hat{Q}_i, Q_j, \hat{\mathfrak{w}}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n),(m,m+1)\\ (\hat{\pi}_i \hat{Q}_i, \hat{Q}_j, \hat{\mathfrak{w}}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n),(m,m)\end{cases}. \tag{27}$$

The contour $\mathsf{C}(\mathfrak{I})$ is positively oriented and encircles the support $\mathfrak{I}$ and 0 as shown in the figure below.



This computation paves the way for asymptotic results related to the growth process. We will now describe a simple such scaling limit which despite its simplicity reveals the richness of the determinantal point process studied here.

We first need to introduce the notion of a discrete ensemble associated to continuous orthogonal polynomials. Special cases of these ensembles and their degenerations including the discrete sine and discrete Bessel kernel have appeared in many problems of random partitions, most famously in the asymptotics of the Plancherel measure [24], [87]. More recently, a connection with the asymmetric simple exclusion process (ASEP) was found in [30]. Such a connection is surprising since ASEP is not expected to be a determinantal point process.

So, suppose $\mathcal{W}(dx)$ is a weight on $\mathbb{R}$ for which the moment problem is determinate. Let $P_k^*(x)$ be the k^{th} orthonormal polynomial with respect to this weight with positive leading coefficient. Consider the following kernel $\mathsf{K}_r^{\mathcal{W}}(i,j)$:

$$\mathsf{K}_r^{\mathcal{W}}(i,j) = \int_r^\infty P_i^*(x)P_j^*(x)\mathcal{W}(dx).$$

It is in fact a projection kernel on a subspace of $l^2(\mathbb{Z}_{\geq 0})$ and thus gives rise to a determinantal point process. It has infinitely many particles and is connected to the corresponding orthogonal polynomial ensemble with weight $\mathcal{W}(dx)$ through "hole probabilities", see [30].

Below we will show how these ensembles appear in a concrete stochastic particle system. In fact, we will provide a multilevel (determinantal) extension of these ensembles.

In order to do that we consider a finite distance from the wall scaling limit. More precisely, suppose we scale time as $t(N) = N\tau$ and the arguments of the kernel as $(\tilde{m}_1(N), \tilde{m}_2(N)) = (\lfloor N\eta \rfloor + m_1, \lfloor N\eta \rfloor + m_2)$ and $(\tilde{n}_1(N), \tilde{n}_2(N)) = (\lfloor N\eta \rfloor + n_1, \lfloor N\eta \rfloor + n_2)$ and let $\alpha = \frac{\eta}{\tau}$. Note that, we do not scale the horizontal positions. Then we have the following theorem whose proof can be found in subsection 5.10.2:

Theorem 0.9.

$$\begin{aligned}\lim_{N\to\infty} \mathcal{K}^{t(N)}\left(\left(\left(\tilde{n}_1(N), \tilde{n}_2(N)\right), i\right), \left(\left(\tilde{m}_1(N), \tilde{m}_2(N)\right), j\right)\right) &= \mathfrak{K}_\alpha\left(\left(\left(n_1, n_2\right), i\right), \left(\left(m_1, m_2\right), j\right)\right) \\ &= \int_{I^-}^{I^+} \left[-\mathbf{1}(x \geq \alpha) + \mathbf{1}\left((n_1, n_2) \geq (m_1, m_2)\right)\right] \bar{\mathcal{P}}_i(x) x^{n_2 - m_2} \tilde{\mathcal{P}}_j(x) d\mathfrak{m}(x).\end{aligned}$$

It is easy to see that if restricted to single levels $\mathfrak{K}_\alpha\left(\left(\left(n, n+1\right), i\right), \left(\left(n, n+1\right), j\right)\right)$ gives rise to the determinantal ensemble with kernel $\mathsf{K}_\alpha^{\mathfrak{w}}(i,j)$ and also $\mathfrak{K}_\alpha\left(\left(\left(n, n\right), i\right), \left(\left(n, n\right), j\right)\right)$ gives rise to the ensemble governed by the kernel $\mathsf{K}_\alpha^{\hat{\mathfrak{w}}}(i,j)$; since conjugation by a function does not alter the correlation functions and thus the determinantal measure.

Finally, observe that as in the Borodin Ferrari model, an inhomogeneous (with position dependent jumps) two species analogue of PushASEP (with at most two particles per site) arises if one looks at the rightmost particles in the interlacing array above. In particular the evolution of the particles $\left(X_1^{(0,1)}(t), X_1^{(1,1)}(t), X_2^{(1,2)}(t), \cdots; t \geq 0\right)$ is autonomous.

Symmetric functions: Karlin-McGregor polynomials We close this introduction with the following remark. Most of the integrable probabilistic systems that have been studied have certain symmetric functions in the background. A very distringuished basis of the algebra of symmetric functions are the Schur polynomials s_ν indexed by partitions ν:

$$s_\nu(z_1, \cdots, z_n) = \frac{\det\left(z_i^{\lambda_j + N - j}\right)_{i,j=1}^N}{\Delta_N(z)}.$$

Through these polynomials Okounkov and later Okounkov and Reshitikhin introduced the Schur measures [115] and Schur processes, see [119] and showed that they form a determinantal point process. Since then many integrable deformations of the Schur functions, originating in algebra, were used, such as Macdonald polynomials $P_\nu(z; q, t)$ which are behind the introduction of Macdonald processes, see [19] and also symmetric functions associated to vertex models, see [18]. However, none of these measures on interlacing arrays retains the determinantal point process structure.

In our setting above another family of symmetric functions which we call Karlin-McGregor polynomials Q_ν, see Section 5.7 of Chapter 5, indexed by partitions ν is responsible for the exact solvability. The informal relation to the Schur polynomials can be seen as follows. Note that, $\psi_k(z) = z^k$ are orthogonal on the circle $\mathbb{T}$ with respect to $\frac{dz}{z}$:

$$\int_{\mathbb{T}} \psi_{k_1}(z)\bar{\psi}_{k_2}(z)\frac{dz}{z} = 2\pi\imath\delta_{k_1,k_2}.$$

Then, with the following correspondence in one dimension,

$$\left(z^k, \frac{dz}{z}, \mathbb{T}\right) \rightsquigarrow (Q_k(x), \mathfrak{w}(dx), \mathfrak{I})$$

we have:

$$s_\nu(z_1, \cdots, z_n) \rightsquigarrow Q_\nu(x_1, \cdots, x_n).$$

In fact all the probabilistic quantities of interest in the model above can be re-expressed in terms of properties such as branching and Pieri rules and eigenfunction relations of these symmetric functions.

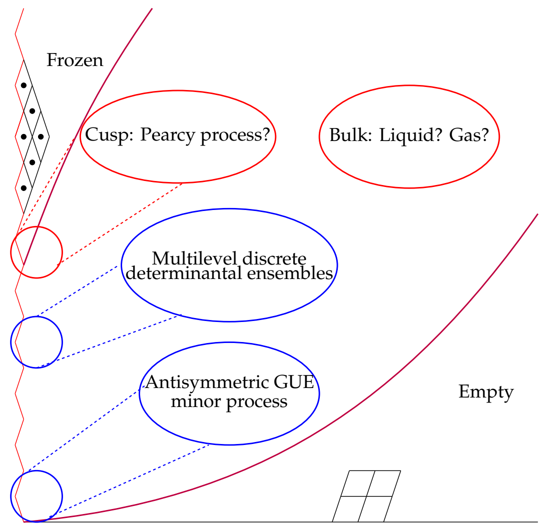
0.6 Outlook and questions

Here, we list some natural questions and directions for research that arise from the investigations of this thesis. A few of these problems are rather open-ended but some of the more concrete ones are works in progress.

1. What is the right, namely integrable, inhomogeneous generalization of the Borodin-Ferrari (2 + 1)-dimensional growth model? This could provide a route to some exact solvability in inhomogeneous TASEP which has thus far resisted many efforts. A particular case is the slow bond problem for which a breakthrough was achieved for the leading order behaviour using non-exactly solvable techniques, see [15].

2. Study different scaling regimes for the inhomogeneous process introduced above. It is clear from simulations, that I have performed, that interesting behaviour arises when one introduces for example slow regions, periodic or trigonometric rates. The analysis of course boils down to the associated one-dimensional othogonal polynomials. Another question is whether in any of the possible scaling regimes perturbations of the rates still give the same asymptotic behaviour. This again will boil down to universality statements for orthogonal polynomials. See the figure below for a caricature

of possible behaviour (the complete picture has been proven in the special models [23], [46])



3. As explained in detail in Chapter 5, one can associate a branching graph to the growth model with a wall, that we call generalized type-BC branching graph, its multiplicities are given by general product form weights. Is it possible, at least for certain multiplicities, to describe its boundary? Moreover, what is the relation of such extreme coherent measures with dynamics on the graph. In the case of both the Gelfand-Tsetlin graph and the type-BC graph there is an exact correspondence with continuous time birth and death chain dynamics, discrete time Bernoulli and also geometric jumps. A more ambitious direction would be to develop some kind of perturbation theory for these graphs.

4. The N-dimensional Hua-Pickrell diffusions give rise to time dependent determinantal point processes. The first question arising is to compute their correlation kernels from both the stationary measure and arbitrary deterministic initial condition and take the $N \to \infty$ scaling limit. Another direction is to construct SDEs for the limiting process, at least for some initial conditions. This along with the original construction will provide, for the first time, the complete picture for infinite dimensional dynamics related to random matrices.

5. Extend the matrix Bougerol identity to the orthogonal and symplectic matrix setting. In one dimension Bougerol's and Dufresne's identities are related by a simple time-change. It is worth investigating whether there is a relation between their matrix versions. This will provide a connection between the inverse Wishart and Hua-Pickrell measures.

6. Construct the β-Laguerre multilevel process and study the particle system at its edge, at least for fixed times.

Chapter 1

Interlacing diffusions

1.1 Introduction

This chapter is a condensed version of [6], which is joint work with Neil O'Connell and Jon Warren. Some of the longer proofs are omitted. However, we explain the strategy of proof along with the key ingredients in Section 1.5.

In this chapter we study in some generality intertwinings and couplings between Karlin-McGregor semigroups (see [94], also [91]) associated with one dimensional diffusion processes and their duals. Let $X(t)$ be a diffusion process with state space an interval $I \subset \mathbb{R}$ with end points $l < r$ and transition density $p_t(x,y)$. We define the Karlin-McGregor semigroup associated with X, with n particles, by its transition densities (with respect to Lebesgue measure) given by,

$$\det(p_t(x_i, y_j))_{i,j=1}^n,$$

for $x, y \in W^n(I^\circ)$ where $W^n(I^\circ) = (x = (x_1, \cdots, x_n) : l < x_1 \leq \cdots \leq x_n < r)$. This sub-Markov semigroup is exactly the semigroup of n independent copies of the diffusion process X which are killed when they intersect. For such a diffusion process $X(t)$ we consider the conjugate (see [152]) or Siegmund dual (see [49] or the original paper [144]) diffusion process $\hat{X}(t)$ via a description of its generator and boundary behaviour in the next subsection. The key relation conjugate diffusion processes satisfy is the following (see Lemma 1.1), with $z, z' \in I^\circ$,

$$\mathbb{P}_z(X(t) \leq z') = \mathbb{P}_{z'}(\hat{X}(t) \geq z).$$

We will obtain *couplings* of h-transforms of Karlin-McGregor semigroups associated with a diffusion process and its conjugate so that the corresponding processes *interlace*. We say that $y \in W^n(I^\circ)$ and $x \in W^{n+1}(I^\circ)$ interlace and denote this by $y \prec x$ if $x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{n+1}$. Note that this defines a space denoted by $W^{n,n+1}(I^\circ) = ((x,y) : l < x_1 \leq y_1 \leq x_2 \leq \cdots \leq x_{n+1} <$

r),

$$\begin{array}{ccccccccc} x_1 & y_1 & x_2 & y_1 & x_3 & & x_n & y_n & x_{n+1} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \end{array},$$

with the following two-level representation,

$$\begin{array}{ccccccccccc} & y_1 & & y_1 & & \cdots\cdots & & & y_n & & \\ & \bullet & & \bullet & & & & & \bullet & & \\ x_1 & & x_2 & & x_3 & & \cdots & x_n & & x_{n+1} & \\ \bullet & & \bullet & & \bullet & & & \bullet & & \bullet & \end{array}.$$

Similarly, we say that $x, y \in W^n(I^\circ)$ interlace if $l < y_1 \leq x_1 \leq y_2 \leq \cdots \leq x_n < r$. Again this defines the space $W^{n,n}(I^\circ) = ((x, y) : l < y_1 \leq x_1 \leq y_2 \leq \cdots \leq x_n < r)$,

$$\begin{array}{cccccccc} y_1 & x_1 & y_2 & x_2 & & x_{n-1} & y_n & x_n \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet \end{array},$$

with the two-level representation,

$$\begin{array}{ccccccccc} y_1 & & y_2 & & \cdots & & y_n & & \\ \bullet & & \bullet & & & & \bullet & & \\ & x_1 & & x_2 & \cdots & x_{n-1} & & x_n & \\ & \bullet & & \bullet & & \bullet & & \bullet & \end{array}.$$

Our starting point are explicit transition kernels, actually arising from the consideration of stochastic coalescing flows. These kernels defined on $W^{n,n+1}(I^\circ)$ (or $W^{n,n}(I^\circ)$) are given in terms of block determinants and give rise to a Markov process $Z = (X, Y)$ with (sub-)Markov transition semigroup Q_t with joint dynamics described as follows. Let L and $\hat{L}$ be the generators of a pair of one dimensional diffusions in Siegmund duality. Then, after an appropriate Doob's h-transformation Y evolves *autonomously* as n $\hat{L}$-diffusions conditioned not to intersect. The X components then evolve as $n + 1$ (or n) independent L-diffusions reflected off the random Y barriers, a notion made precise in the next subsection. Our main result, Theorem 1.15 in the text, states (modulo technical assumptions) that under a special initial condition for $Z = (X, Y)$, the *non-autonomous* X component is distributed as a Markov process in its own right. Its evolution governed by an explicit Doob's h-transform of the Karlin-McGregor semigroup associated with $n + 1$ (or n) L-diffusions.

At the heart of this result lie certain intertwining relations, obtained immediately from the special structure of Q_t, of the form,

$$P_t\Lambda = \Lambda Q_t\,, \tag{1.1}$$

$$\Pi\hat{P}_t = Q_t\Pi\,, \tag{1.2}$$

where Λ is an explicit positive kernel (not yet normalized), Π is the operator induced by the projection on the Y level, P_t is the Karlin-McGregor semigroup associated with the one dimensional diffusion process with transition density $p_t(x, y)$ and $\hat{P}_t$ the corresponding semigroup associated with its conjugate (some conditions and more care is needed regarding boundary behaviour for which the reader is referred to the next section).

Now we move towards building a multilevel process. First, note that by concatenating $W^{1,2}(I^\circ), W^{2,3}(I^\circ), \cdots, W^{N-1,N}(I^\circ)$ we obtain the space of (continuous) Gelfand-Tsetlin patterns of depth N denoted by $\mathbb{GT}_c(N)$,

$$\mathbb{GT}_c(N) = \{(X^{(1)}, \cdots, X^{(N)}) : X^{(n)} \in W^n(I^\circ),\ X^{(n)} \prec X^{(n+1)}\}\,.$$

A point $(X^{(1)}, \cdots, X^{(N)}) \in \mathbb{GT}_c(N)$ is typically depicted as an array as shown in the following diagram:

$$\begin{array}{cccccccccc}
 & & & & X_1^{(1)} & & & & & \\
 & & & X_1^{(2)} & & X_2^{(2)} & & & & \\
 & & X_1^{(3)} & & X_2^{(3)} & & X_3^{(3)} & & & \\
 & \iddots & & & \vdots & & & \ddots & & \\
X_1^{(N)} & X_2^{(N)} & X_3^{(N)} & & \cdots\cdots & & & & X_{N-1}^{(N)} & X_N^{(N)}
\end{array}$$

Similarly, by concatenating $W^{1,1}(I^\circ), W^{1,2}(I^\circ), W^{2,2}(I^\circ), \cdots, W^{N,N}(I^\circ)$ we obtain the space of (continuous) symplectic Gelfand-Tsetlin pattern of depth N denoted by $\mathbb{GT}_{c,\mathbf{s}}(N)$,

$$\mathbb{GT}_{c,\mathbf{s}}(N) = \{(X^{(1)}, \hat{X}^{(1)} \cdots, X^{(N)}, \hat{X}^{(N)}) : X^{(n)}, \hat{X}^{(n)} \in W^n(I^\circ),\ X^{(n)} \prec \hat{X}^{(n)} \prec X^{(n+1)}\}\,,$$

$$\begin{array}{|cccccccc}
X_1^{(1)} & & & & & & & \\
 & \hat{X}_1^{(1)} & & & & & & \\
X_1^{(2)} & & X_2^{(2)} & & & & & \\
 & \hat{X}_1^{(2)} & & \hat{X}_2^{(2)} & & & & \\
X_1^{(3)} & & X_2^{(3)} & & X_3^{(3)} & & & \\
\vdots & & \vdots & & & \ddots & & \\
X_1^{(N)} & & X_2^{(N)} & \cdots & & & X_N^{(N)} & \\
 & \hat{X}_1^{(N)} & & \hat{X}_2^{(N)} & \cdots & \hat{X}_{N-1}^{(N)} & & \hat{X}_N^{(N)}
\end{array}$$

Theorem 1.15 allows us to concatenate a sequence of $W^{n,n+1}$-valued processes (or two-level processes), by a procedure described at the beginning of Section 3, in order to build diffusion processes in the space of (continuous) Gelfand-Tsetlin patterns so that each level is Markovian with explicit transition densities.

Such examples of dynamics on *discrete* Gelfand-Tsetlin patterns have been extensively studied over the past decade as models for random surface growth, see in particular [23], [21], [47]. They have also been considered in relation to building infinite dimensional Markov processes, preserving some distinguished measures of representation theoretic origin, on the boundary of these Gelfand-Tsetlin graphs via the *method of intertwiners*; see Borodin and Olshanski [28] for the type A case and more recently Cuenca [50] for the type BC. We pursue both of these directions in detail in Chapter 5.

Returning to the continuum discussion many old and new examples fit into the framework developed in this chapter. Here, we will restrict ourselves to presenting in Section 1.3 some simple cases related to the three classical unitarily invariant random matrix ensembles: the Gaussian (GUE), the Laguerre (LUE) and Jacobi (JUE). We will moreover provide a construction related to the Hua-Pickrell measures in Section 3.6 of Chapter 3. For more examples and in particular for a study of diffusion processes with discrete spectrum and connections to the theory of total positivity (see the classical monograph of Karlin [91]) the reader is referred to [6].

We now mention a couple of recent works in the literature that are related to the study undertaken here. Firstly a different approach based on generators for obtaining couplings of intertwined multidimensional diffusion processes via hard reflection is investigated in Theorem 3 of [125]. This has subsequently been extended by Sun [149] to isotropic diffusion coefficients, who making use of this has independently obtained similar results to ours in the case of the multilevel *LUE* and *JUE* processes. Moreover, a general β extension of the intertwining relations for the random matrix related aforementioned processes is established in the next chapter. Also, some results from this chapter are used in Chapter 3 to construct an infinite dimensional Feller process on the so called *graph of spectra*, that is the continuum analogue of the Gelfand-Tsetlin graph, which leaves the celebrated Hua-Pickrell measures invariant.

Finally, we study the interacting particle systems with one-sided collisions at either edge of such Gelfand-Tsetlin pattern valued processes and give explicit Schutz-type determinantal transition densities for them in terms of derivatives and integrals of the one dimensional kernels. This also leads to formulas for the largest and smallest eigenvalues of the *LUE* and *JUE* ensembles in analogy to the ones obtained in [164] for the *GUE*.

1.2 Two-level construction

1.2.1 Set up of conjugate diffusions

Since our basic building blocks will be one dimensional diffusion processes and their conjugates we introduce them here and collect a number of facts about them (for justifications and proofs see the Appendix). The majority of the facts below can be found in the seminal book of Ito and McKean [83], and also more specifically regarding the transition densities of general one dimensional diffusion processes, in the classical paper of McKean [106] and also section 4.11 of [83] which we partly follow at various places.

We consider $(X_t)_{t\geq 0}$ a time homogeneous one dimensional diffusion process with state space an interval I with endpoints $l < r$ which can be open or closed, finite or infinite (interior denoted by I°) with infinitesimal generator given by,

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx},$$

with domain to be specified later in this section. We assume that $a \in C^1(I^\circ)$ with $a(x) > 0$ for $x \in I^\circ$ and $b(x) \in C(I^\circ)$. In order to be more concise, we will frequently refer to such a diffusion process with generator L as an L-diffusion.

We start by giving the very convenient description of the generator L in terms of its speed measure and scale function. Define its scale function $s(x)$ by $s'(x) = exp\Big(-\int_c^x \frac{b(y)}{a(y)}dy\Big)$ (the scale function is defined up to affine transformations) where c is an arbitrary point in I°, its speed measure with density $m(x) = \frac{1}{s'(x)a(x)}$ in I° with respect to the Lebesgue measure (note that it is a Radon measure in I° and also strictly positive in I°) and speed function $M(x) = \int_c^x m(y)dy$. With these definitions the formal infinitesimal generator L can be written as,

$$L = \mathcal{D}_m \mathcal{D}_s \ ,$$

where $\mathcal{D}_m = \frac{1}{m(x)}\frac{d}{dx} = \frac{d}{dM}$ and $\mathcal{D}_s = \frac{1}{s'(x)}\frac{d}{dx} = \frac{d}{ds}$.

We now define the conjugate diffusion (see [152]) or Siegmund dual (see [144]) $(\hat{X}_t)_{t\geq 0}$ of X to be a diffusion process with generator,

$$\hat{L} = a(x)\frac{d^2}{dx^2} + (a'(x) - b(x))\frac{d}{dx},$$

and domain to be given shortly.

The following relations are easy to verify and are key to us.

$$\hat{s}'(x) = m(x) \ and \ \hat{m}(x) = s'(x).$$

So the conjugation operation swaps the scale functions and speed measures. In particular

$$\hat{L} = \mathcal{D}_{\hat{m}} \mathcal{D}_{\hat{s}} = \mathcal{D}_s \mathcal{D}_m \ .$$

Using Feller's classification of boundary points (see Appendix of [6] for example) we obtain the following table for the boundary behaviour of the diffusion processes with generators L and $\hat{L}$ at l or r,

Bound. Class. of L	Bound. Class. of $\hat{L}$
natural	natural
entrance	exit
exit	entrance
regular	regular

We briefly explain what these boundary behaviours mean. A process can neither be started at, nor reach in finite time a *natural* boundary point. It can be started from an *entrance* point but such a boundary point cannot be reached from the interior I°. Such points are called *inaccessible* and can be removed from the state space. A diffusion can reach an *exit* boundary point from I° and once it does it is absorbed there. Finally, at a *regular* (also called entrance and exit) boundary point a variety of behaviours is possible and we need to *specify* one such. We will only be concerned with the two extreme possibilities namely *instantaneous reflection* and *absorption* (sticky behaviour interpolates between the two and is not considered here). Furthermore, note that if l is *instantaneously reflecting* then (see for example Chapter 2 paragraph 7 in [36]) $Leb\{t : X_t = l\} = 0$ *a.s.* and analogously for the upper boundary point r.

Now in order to describe the domain, $Dom(L)$, of the diffusion process with formal generator L we first define the following function spaces (with the obvious abbreviations),

$$\begin{aligned}
C(\bar{I}) &= \{f \in C(I^\circ) : \lim_{x \downarrow l} f(x), \lim_{x \uparrow r} f(x) \text{ exist and are finite}\} \ , \\
\mathfrak{D} &= \{f \in C(\bar{I}) \cap C^2(I^\circ) : Lf \in C(\bar{I})\} \ , \\
\mathfrak{D}_{nat} &= \mathfrak{D} \ , \\
\mathfrak{D}_{entr} &= \mathfrak{D}_{refl} = \{f \in \mathfrak{D} : (\mathcal{D}_s f)(l^+) = 0\} \ , \\
\mathfrak{D}_{exit} &= \mathfrak{D}_{abs} = \{f \in \mathfrak{D} : (Lf)(l^+) = 0\}.
\end{aligned}$$

Similarly, define $\mathfrak{D}^{nat}, \mathfrak{D}^{entr}, \mathfrak{D}^{refl}, \mathfrak{D}^{exit}, \mathfrak{D}^{abs}$ by replacing l with r in the definitions above. Then the domain of the generator of the $(X_t)_{t\geq 0}$ diffusion process (with generator L) with boundary behaviour i at l and j at r where $i, j \in \{nat, entr, refl, exit, abs\}$ is given by,

$$Dom(L) = \mathfrak{D}_i \cap \mathfrak{D}^j \ .$$

For justifications see for example Chapter 8 in [68] and for an entrance boundary point also Theorem 12.2 of [96] or page 122 of [106].

Coming back to conjugate diffusions note that the boundary behaviour of X_t, the L-diffusion, determines the boundary behaviour of $\hat{X}_t$, the $\hat{L}$-diffusion, except at a regular point. At such a point we define the boundary behaviour of the $\hat{L}$-diffusion to be dual to that of the L-diffusion. Namely, if l is regular reflecting for L then we define it to be regular absorbing for $\hat{L}$. Similarly, if l is regular absorbing for L we define it to be regular reflecting for $\hat{L}$. The analogous definition being enforced at the upper boundary point r. Furthermore, we denote the semigroups associated with X_t and $\hat{X}_t$ by P_t and $\hat{\mathsf{P}}_t$ respectively and note that $\mathsf{P}_t 1 = \hat{\mathsf{P}}_t 1 = 1$. We remark that at an *exit* or *regular absorbing* boundary point the transition *kernel* $p_t(x, dy)$ associated with P_t has an *atom* there with mass (depending on t and x) the probability that the diffusion has reached that point by time t started from x.

We finally arrive at the following duality relation, going back in some form to Siegmund. This is proven via an approximation by birth and death chains in Section 4 of [49]. A proof, following [165] (where the proof is given in a special case), can be found in the Appendix of [6]. The reader should note the restriction to the interior I°.

Lemma 1.1. $\mathsf{P}_t \mathbf{1}_{[l,y]}(x) = \hat{\mathsf{P}}_t \mathbf{1}_{[x,r]}(y)$ *for* $x, y \in I^\circ$.

Now, it is well known that, the transition density $p_t(x, y) : (0, \infty) \times I^\circ \times I^\circ \to (0, \infty)$

of any one dimensional diffusion process with a speed measure which has a continuous density with respect to the Lebesgue measure in I° (as is the case in our setting) is continuous in (t,x,y). Moreover, under our assumptions $\partial_x p_t(x,y)$ exists for $x \in I^\circ$ and as a function of (t,y) is continuous in $(0,\infty)\times I^\circ$ (see Theorem 4.3 of [106]).

This fact along with Lemma 1.1 gives the following relationships between the transition densities for $x,y \in I^\circ$,

$$p_t(x,y) = \partial_y \hat{\mathsf{P}}_t \mathbf{1}_{[x,r]}(y) = \partial_y \int_x^r \hat{p}_t(y,dz)\,, \tag{1.3}$$

$$\hat{p}_t(x,y) = -\partial_y \mathsf{P}_t \mathbf{1}_{[l,x]}(y) = -\partial_y \int_l^x p_t(y,dz). \tag{1.4}$$

Before closing this section, we note that the speed measure is the *symmetrizing* measure of the diffusion process and this shall be useful in what follows. In particular, for $x,y \in I^\circ$ we have,

$$\frac{m(y)}{m(x)} p_t(y,x) = p_t(x,y). \tag{1.5}$$

1.2.2 Transition kernels for two-level processes

First, we recall the definitions of the interlacing spaces our processes will take values in,

$$\begin{aligned} W^n(I^\circ) &= ((x) : l < x_1 \le \cdots \le x_n < r)\,, \\ W^{n,n+1}(I^\circ) &= ((x,y) : l < x_1 \le y_1 \le x_2 \le \cdots \le x_{n+1} < r)\,, \\ W^{n,n}(I^\circ) &= ((x,y) : l < y_1 \le x_1 \le y_2 \le \cdots \le x_n < r)\,, \\ W^{n+1,n}(I^\circ) &= ((x,y) : l < y_1 \le x_1 \le y_2 \le \cdots \le y_{n+1} < r). \end{aligned}$$

Also define for $x \in W^n(I^\circ)$,

$$W^{\bullet,n}(x) = \{y \in W^{\bullet}(I^\circ) : (x,y) \in W^{\bullet,n}(I^\circ)\}.$$

Boundary behaviour assumption We now make the following standing assumption, enforced throughout the chapter, on the boundary behaviour of the one dimensional diffusion process with generator L, depending on which interlacing space our two-level process defined next takes values in. Its significance will be explained later on. Note that any possible combination is allowed between the behaviour at l and r.

$\boldsymbol{W^{n,n+1}(I^\circ)}$

$$l \text{ is either } \textit{Natural} \text{ or } \textit{Entrance} \text{ or } \textit{Regular Reflecting}\,, \tag{1.6}$$

$$r \text{ is either } \textit{Natural} \text{ or } \textit{Entrance} \text{ or } \textit{Regular Reflecting}. \tag{1.7}$$

$\boldsymbol{W^{n,n}(I^\circ)}$

$$l \text{ is either } \textit{Natural} \text{ or } \textit{Exit} \text{ or } \textit{Regular Absorbing}\,, \tag{1.8}$$

$$r \text{ is either } \textit{Natural} \text{ or } \textit{Entrance} \text{ or } \textit{Regular Reflecting}. \tag{1.9}$$

$\boldsymbol{W^{n+1,n}(I^\circ)}$

$$l \text{ is either } \textit{Natural} \text{ or } \textit{Exit} \text{ or } \textit{Regular Absorbing}\,, \tag{1.10}$$

$$r \text{ is either } \textit{Natural} \text{ or } \textit{Exit} \text{ or } \textit{Regular Absorbing}. \tag{1.11}$$

Coalescing diffusions We shall begin by considering the following stochastic process which we will denote by $(\mathbf{\Phi}_{0,t}(x_1), \cdots, \mathbf{\Phi}_{0,t}(x_n); t \geq 0)$. It consists of a system of n independent L-diffusions started from $x_1 \leq \cdots \leq x_n$ which *coalesce* and move together once they meet. This is a process in $W^n(I)$ which once it reaches any of the hyperplanes $\{x_i = x_{i+1}\}$ continues there forever. We have the following proposition for the finite dimensional distributions of the coalescing process,

Proposition 1.2. *For $z, z' \in W^n(I^\circ)$,*

$$\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(z_i) \leq z_i' \text{ for } 1 \leq i \leq n\Big) = \det\Big(\mathsf{P}_t \mathbf{1}_{[l,z_j']}(z_i) - \mathbf{1}(i<j)\Big)_{i,j=1}^n .$$

Proof. This is done for Brownian motions in Proposition 9 of [164] using a generic argument based on continuous non-intersecting paths. The only variation here is that there might be an atom at l which however does not alter the proof. □

We now define the kernel $q_t^{n,n+1}((x,y),(x',y'))dx'dy'$ on $W^{n,n+1}(I^\circ)$ as follows,

Definition 1.3. *For $(x,y),(x',y') \in W^{n,n+1}(I^\circ)$ define $q_t^{n,n+1}((x,y),(x',y'))$ by,*

$$q_t^{n,n+1}((x,y),(x',y')) =$$
$$= \frac{\prod_{i=1}^n \hat{m}(y_i')}{\prod_{i=1}^n \hat{m}(y_i)} (-1)^n \frac{\partial^n}{\partial_{y_1} \cdots \partial_{y_n}} \frac{\partial^{n+1}}{\partial_{x_1'} \cdots \partial_{x_{n+1}'}} \mathbb{P}\Big(\mathbf{\Phi}_{0,t}(x_i) \leq x_i', \mathbf{\Phi}_{0,t}(y_j) \leq y_j' \text{ for all } i,j\Big) .$$

This density exists by virtue of the regularity of the one dimensional transition densities. It is then an elementary computation using Proposition 1.2 and Lemma 1.1, along with relation (1.4), that $q_t^{n,n+1}$ can be written out explicitly as shown below. Note that each y_i and x_j' variable appears only in a certain row or column respectively.

$$q_t^{n,n+1}((x,y),(x',y')) = \det \begin{pmatrix} A_t(x,x') & B_t(x,y') \\ C_t(y,x') & D_t(y,y') \end{pmatrix} \tag{1.12}$$

where,

$$
\begin{aligned}
A_t(x,x')_{ij} &= \partial_{x'_j}\mathsf{P}_t\mathbf{1}_{[l,x'_j]}(x_i) = p_t(x_i,x'_j)\,,\\
B_t(x,y')_{ij} &= \hat{m}(y'_j)(\mathsf{P}_t\mathbf{1}_{[l,y'_j]}(x_i)-\mathbf{1}(j\geq i))\,,\\
C_t(y,x')_{ij} &= -\hat{m}^{-1}(y_i)\partial_{y_i}\partial_{x'_j}\mathsf{P}_t\mathbf{1}_{[l,x'_j]}(y_i) = -\mathcal{D}_s^{y_i}p_t(y_i,x'_j)\,,\\
D_t(y,y')_{ij} &= -\frac{\hat{m}(y'_j)}{\hat{m}(y_i)}\partial_{y_i}\mathsf{P}_t\mathbf{1}_{[l,y'_j]}(y_i) = \hat{p}_t(y_i,y'_j).
\end{aligned}
$$

We now define for $t>0$ the operators $Q_t^{n,n+1}$ acting on the bounded Borel functions on $W^{n,n+1}(I^\circ)$ by,

$$
(Q_t^{n,n+1}f)(x,y) = \int_{W^{n,n+1}(I^\circ)} q_t^{n,n+1}((x,y),(x',y'))f(x',y')dx'dy'. \tag{1.13}
$$

Then the following facts hold:

Lemma 1.4.

$$
\begin{aligned}
&Q_t^{n,n+1}1 \leq 1,\\
&Q_t^{n,n+1}f \geq 0 \text{ for } f\geq 0.
\end{aligned}
$$

Proof. The first property follows from performing the dx' integration (which is easily done by the very structure of the entries of $q_t^{n,n+1}$) first in equation (1.13) and then we are left with the integral,

$$
\int_{W^n(I^\circ)} \det(\hat{p}_t(y_i,y'_j))_{i,j}^n dy' \leq 1.
$$

The *positivity* preserving property also follows immediately from the original definition, since $\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(x_i)\leq x'_i, \mathbf{\Phi}_{0,t}(y_j)\leq y'_j \text{ for all } i,j\Big)$ is increasing in the x'_i and decreasing in the y_i respectively. ◻

In fact, $Q_t^{n,n+1}$ defined above, forms a sub-Markov semigroup, associated with a Markov process $Z=(X,Y)$, with possibly finite lifetime, described informally as follows: the X components follow independent L-diffusions reflected off the Y components. More precisely assume that the L-diffusion is given as the pathwise unique solution X to the *SDE*,

$$
d\mathsf{X}(t) = \sqrt{2a(\mathsf{X}(t))}d\beta(t) + b(\mathsf{X}(t))dt + dK^l(t) - dK^r(t)
$$

where β is a standard Brownian motion and K^l and K^r are (possibly zero) positive finite variation processes that only increase when $\mathsf{X}=l$ or $\mathsf{X}=r$, so that $\mathsf{X}\in I$ and $Leb\{t:\mathsf{X}(t)=l \text{ or } r\}=0$ a.s. We write $\mathfrak{s}_L$ for the corresponding measurable solution map on path space, namely so that $\mathsf{X}=\mathfrak{s}_L(\beta)$.

Consider the following system of *SDEs* with reflection in $W^{n,n+1}$ which can be described in words as follows. The Y components evolve as n autonomous $\hat{L}$-diffusions stopped when they collide or when (if) they hit l or r, and we denote this time by $T^{n,n+1}$. The X components evolve as $n+1$ L-diffusions reflected off the Y particles.

$$
\begin{aligned}
dX_1(t) &= \sqrt{2a(X_1(t))}d\beta_1(t) + b(X_1(t))dt + dK^l(t) - dK_1^+(t), \\
dY_1(t) &= \sqrt{2a(Y_1(t))}d\gamma_1(t) + (a'(Y_1(t)) - b(Y_1(t)))dt, \\
dX_2(t) &= \sqrt{2a(X_2(t))}d\beta_2(t) + b(X_2(t))dt + dK_2^-(t) - dK_2^+(t), \\
&\vdots \\
dY_n(t) &= \sqrt{2a(Y_n(t))}d\gamma_n(t) + (a'(Y_n(t)) - b(Y_n(t)))dt, \\
dX_{n+1}(t) &= \sqrt{2a(X_{n+1}(t))}d\beta_{n+1}(t) + b(X_{n+1}(t))dt + dK_{n+1}^-(t) - dK^r(t).
\end{aligned}
\tag{1.14}
$$

Here $\beta_1, \cdots, \beta_{n+1}, \gamma_1, \cdots, \gamma_n$ are independent standard Brownian motions and the positive finite variation processes K^l, K^r, K_i^+, K_i^- are such that K^l (possibly zero) increases only when $X_1 = l$, K^r (possibly zero) increases only when $X_{n+1} = r$, $K_i^+(t)$ increases only when $Y_i = X_i$ and $K_i^-(t)$ only when $Y_{i-1} = X_i$, so that $(X_1(t) \le Y_1(t) \le \cdots \le X_{n+1}(t); t \ge 0) \in W^{n,n+1}(I)$ up to time $T^{n,n+1}$. Note that, X either reflects at l or r or does not visit them at all by our boundary conditions (1.6) and (1.7). The problematic possibility of an X component being trapped between a Y particle and a boundary point and pushed in opposite directions does not arise, since the whole process is then instantly stopped.

These *SDEs* are well-posed, so that in particular (X, Y) is Markovian, under a Yamada-Watanabe condition, see Proposition 1.21. Moreover, by virtue of the following result these *SDEs* provide a precise description of the dynamics of the two-level process $Z = (X, Y)$ associated with $Q_t^{n,n+1}$.

Proposition 1.5. *Under the standing assumption of Subsection 1.5.2 along with the assumptions of Proposition 1.22 or Proposition 1.24 we have that $Q_t^{n,n+1}$ is the sub-Markov semigroup associated with the (Markovian) system of SDEs (1.14) in the sense that if $\mathbf{Q}_{x,y}^{n,n+1}$ governs the processes (X, Y) satisfying the SDEs (1.14) and with initial condition (x, y) then for any f continuous with compact support and fixed $T > 0$,*

$$
\int_{W^{n,n+1}(I^\circ)} q_T^{n,n+1}((x,y),(x',y'))f(x',y')dx'dy' = \mathbf{Q}_{x,y}^{n,n+1}\left[f(X(T),Y(T))\mathbf{1}(T < T^{n,n+1})\right].
$$

For further motivation regarding the definition of $Q_t^{n,n+1}$ and moreover, a completely different argument for its semigroup property, that however does not describe explicitly the dynamics of X and Y, we refer the reader to the next subsection 1.2.3.

We now briefly study some properties of $Q_t^{n,n+1}$, that are immediate from its algebraic structure (with no reference to the *SDEs* above required). In order to proceed and fix notations for the rest of this section, start by defining the Karlin-McGregor semigroup P_t^n

associated with n L-diffusions in I° given by the transition density, with $x, y \in W^n(I^\circ)$,

$$p_t^n(x,y)dy = \det(p_t(x_i,y_j))_{i,j=1}^n dy. \tag{1.15}$$

Note that, in the case an exit or regular absorbing boundary point exists, P_t^1 is the semigroup of the L-diffusion *killed* and not absorbed at that point. In particular it is not the same as P_t which is a Markov semigroup. Similarly, define the Karlin-McGregor semigroup $\hat{P}_t^n$ associated with n $\hat{L}$-diffusions by,

$$\hat{p}_t^n(x,y)dy = \det(\hat{p}_t(x_i,y_j))_{i,j=1}^n dy, \tag{1.16}$$

with $x, y \in W^n(I^\circ)$. The same comment regarding absorbing and exit boundary points applies here as well.

Now, define the operators $\Pi_{n,n+1}$, induced by the projections on the Y level as follows with f a bounded Borel function on $W^n(I^\circ)$,

$$(\Pi_{n,n+1}f)(x,y) = f(y).$$

The following proposition immediately follows by performing the dx' integration in the explicit formula for the block determinant (as already implied in the proof that $Q_t^{n,n+1}1 \le 1$).

Proposition 1.6. *For $t > 0$ and f a bounded Borel function on $W^n(I^\circ)$ we have,*

$$\Pi_{n,n+1}\hat{P}_t^n f = Q_t^{n,n+1}\Pi_{n,n+1}f. \tag{1.17}$$

We also record here the probabilistic consequences of the proposition above. The intertwining relation (1.17), being an instance of Dynkin's criterion (see for example Exercise 1.17 Chapter 3 of [134]), implies that the evolution of Y is Markovian with respect to the joint filtration of X and Y i.e. of the process Z and we take this as the definition of Y being autonomous. Moreover, Y is distributed as n $\hat{L}$-diffusions killed when they collide or when they hit l or r. In summary, the Y components form an *autonomous diffusion* process. Finally, by taking $f \equiv 1$ above we get that the finite lifetime of Z exactly corresponds to the killing time of Y, which we denote by $T^{n,n+1}$.

Similarly, we define the kernel $q_t^{n,n}((x,y),(x',y'))dx'dy'$ on $W^{n,n}(I^\circ)$ as follows,

Definition 1.7. *For $(x,y),(x',y') \in W^{n,n}(I^\circ)$ define $q_t^{n,n}((x,y),(x',y'))$ by,*

$$\begin{aligned}
&q_t^{n,n}((x,y),(x',y')) = \\
&= \frac{\prod_{i=1}^n \hat{m}(y_i')}{\prod_{i=1}^n \hat{m}(y_i)}(-1)^n \frac{\partial^n}{\partial_{y_1}\cdots\partial_{y_n}}\frac{\partial^n}{\partial_{x_1'}\cdots\partial_{x_n'}}\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(x_i) \le x_i', \mathbf{\Phi}_{0,t}(y_j) \le y_j' \ \textit{for all} \ \ i,j\Big).
\end{aligned}$$

We note that as before $q_t^{n,n}$ can in fact be written out explicitly,

$$q_t^{n,n}((x,y),(x',y')) = \det \begin{pmatrix} A_t(x,x') & B_t(x,y') \\ C_t(y,x') & D_t(y,y') \end{pmatrix}. \tag{1.18}$$

where,

$$\begin{aligned} A_t(x,x')_{ij} &= \partial_{x'_j} \mathsf{P}_t \mathbf{1}_{[l,x'_j]}(x_i) = p_t(x_i, x'_j), \\ B_t(x,y')_{ij} &= \hat{m}(y'_j)(\mathsf{P}_t \mathbf{1}_{[l,y'_j]}(x_i) - \mathbf{1}(j > i)), \\ C_t(y,x')_{ij} &= -\hat{m}^{-1}(y_i)\partial_{y_i}\partial_{x'_j} \mathsf{P}_t \mathbf{1}_{[l,x'_j]}(y_i) = -\mathcal{D}_s^{y_i} p_t(y_i, x'_j), \\ D_t(y,y')_{ij} &= -\frac{\hat{m}(y'_j)}{\hat{m}(y_i)} \partial_{y_i} \mathsf{P}_t \mathbf{1}_{[l,y'_j]}(y_i) = \hat{p}_t(y_i, y'_j). \end{aligned}$$

Remark 1.8. *Comparing with the $q_t^{n,n+1}$ formulae everything is the same except for the indicator function being $\mathbf{1}(j > i)$ instead of $\mathbf{1}(j \geq i)$.*

Define the operator $Q_t^{n,n}$ for $t > 0$ acting on bounded Borel functions on $W^{n,n}(I^\circ)$ by,

$$(Q_t^{n,n} f)(x,y) = \int_{W^{n,n}(I^\circ)} q_t^{n,n}((x,y),(x',y')) f(x',y') dx' dy'. \tag{1.19}$$

Then with the analogous considerations as for $Q_t^{n,n+1}$ (see subsection 1.2.3 as well), we can see that $Q_t^{n,n}$ should form a sub-Markov semigroup, to which we can associate a Markov process Z, with possibly finite lifetime, taking values in $W^{n,n}(I^\circ)$, the evolution of which we now make precise.

To proceed as before, we assume that the L-diffusion is given by an *SDE* and we consider the following system of *SDEs* with reflection in $W^{n,n}$ which can be described as follows. The Y components evolve as n autonomous $\hat{L}$-diffusions killed when they collide or when (if) they hit the boundary point r, a time which we denote by $T^{n,n}$. The X components evolve as n L-diffusions being kept apart by hard reflection on the Y particles.

$$\begin{aligned} dY_1(t) &= \sqrt{2a(Y_1(t))} d\gamma_1(t) + (a'(Y_1(t)) - b(Y_1(t)))dt + dK^l(t), \\ dX_1(t) &= \sqrt{2a(X_1(t))} d\beta_1(t) + b(X_1(t))dt + dK_1^+(t) - dK_1^-(t), \\ &\vdots \\ dY_n(t) &= \sqrt{2a(Y_n(t))} d\gamma_n(t) + (a'(Y_n(t)) - b(Y_n(t)))dt, \\ dX_n(t) &= \sqrt{2a(X_n(t))} d\beta_n(t) + b(X_n(t))dt + dK_n^+(t) - dK^r(t). \end{aligned} \tag{1.20}$$

Here $\beta_1, \cdots, \beta_n, \gamma_1, \cdots, \gamma_n$ are independent standard Brownian motions and the positive finite variation processes K^l, K^r, K_i^+, K_i^- are such that $\bar{K}^l$ (possibly zero) increases only when $Y_1 = l$, K^r (possibly zero) increases only when $X_n = r$, $K_i^+(t)$ increases only when $Y_i = X_i$ and $K_i^-(t)$ only when $Y_{i-1} = X_i$, so that $(Y_1(t) \leq X_1(t) \leq \cdots \leq X_n(t); t \geq 0) \in W^{n,n}(I)$ up to $T^{n,n}$.

Note that, Y reflects at the boundary point l or does not visit it all and similarly X reflects at r or does not reach it all by our boundary assumptions (1.8) and (1.9). The intuitively problematic issue of Y_n pushing X_n upwards at r does not arise since the whole process is stopped at such instance.

Again, for the fact that these *SDEs* are well-posed, so that in particular (X, Y) is Markovian, under a Yamada-Watanabe condition, see Proposition 1.21. As before, we have the following precise description of the dynamics of the two-level process $Z = (X, Y)$ associated with $Q_t^{n,n}$.

Proposition 1.9. *Under the standing assumption of Subsection 1.5.2 along with the assumptions of Proposition 1.23 we have that $Q_t^{n,n}$ is the sub-Markov semigroup associated with the (Markovian) system of SDEs (1.20) in the sense that if $\mathbf{Q}_{x,y}^{n,n}$ governs the processes (X, Y) satisfying the SDEs (1.20) with initial condition (x, y) then for any f continuous with compact support and fixed $T > 0$,*

$$\int_{W^{n,n}(I^\circ)} q_T^{n,n}((x,y),(x',y'))f(x',y')dx'dy' = \mathbf{Q}_{x,y}^{n,n}\Big[f(X(T),Y(T))\mathbf{1}(T < T^{n,n})\Big].$$

We also define, analogously to before, an operator $\Pi_{n,n}$, induced by the projection on the Y level by,

$$(\Pi_{n,n}f)(x,y) = f(y).$$

We have the following proposition which immediately follows by performing the dx' integration in equation (1.19),

Proposition 1.10. *For $t > 0$ and f a bounded Borel function on $W^n(I^\circ)$ we have,*

$$\Pi_{n,n}\hat{P}_t^n f = Q_t^{n,n}\Pi_{n,n}f. \tag{1.21}$$

This, again implies that the evolution of Y is Markovian with respect to the joint filtration of X and Y. Furthermore, Y is distributed as n $\hat{L}$-diffusions killed when they collide or when (if) they hit the boundary point r (note the difference here to $W^{n,n+1}$ is because of the asymmetry between X and Y and our standing assumption (1.8) and (1.9)). Hence, the Y components form a *diffusion* process and they are *autonomous*. The finite lifetime of Z analogously to before (by taking $f \equiv 1$ in the proposition above), exactly corresponds to the killing time of Y which we denote by $T^{n,n}$.

Finally, we can define the kernel $q_t^{n+1,n}((x,y),(x',y'))dx'dy'$ on $W^{n+1,n}(I^\circ)$ in an analogous way and also the operator $Q_t^{n+1,n}$ for $t > 0$ acting on bounded Borel functions on $W^{n+1,n}(I^\circ)$ as well. The description of the associated process Z in $W^{n+1,n}(I^\circ)$ in words is as follows. The Y components evolve as $n+1$ autonomous $\hat{L}$-diffusions killed when they collide (by our boundary conditions (1.10) and (1.11) if the Y particles do visit l or r they are reflecting there) and the X components evolve as n L-diffusions reflected on the Y particles. These dynamics can be described in terms of *SDEs* with reflection under completely analogous assumptions. The details are omitted.

1.2.3 Stochastic coalescing flow interpretation

The definition of $q_t^{n,n+1}$, and similarly of $q_t^{n,n}$, might look rather mysterious and surprising. It is originally motivated from considering stochastic coalescing flows. Briefly, the finite system $(\mathbf{\Phi}_{0,t}(x_1), \cdots, \mathbf{\Phi}_{0,t}(x_n); t \geq 0)$ can be extended to an infinite system of coalescing L-diffusions starting from each space time point and denoted by $(\mathbf{\Phi}_{s,t}(\cdot), s \leq t)$. This is well documented in Theorem 4.1 of [102] for example. The random family of maps $(\mathbf{\Phi}_{s,t}, s \leq t)$ from I to I enjoys among others the following natural looking and intuitive properties: the *cocycle* or *flow* property $\mathbf{\Phi}_{t_1,t_3} = \mathbf{\Phi}_{t_2,t_3} \circ \mathbf{\Phi}_{t_1,t_2}$, *independence of its increments* $\mathbf{\Phi}_{t_1,t_2} \perp \mathbf{\Phi}_{t_3,t_4}$ for $t_2 \leq t_3$ and *stationarity* $\mathbf{\Phi}_{t_1,t_2} \overset{law}{=} \mathbf{\Phi}_{0,t_2-t_1}$. Finally, we can consider its generalized inverse by $\mathbf{\Phi}_{s,t}^{-1}(x) = \sup\{w : \mathbf{\Phi}_{s,t}(w) \leq x\}$ which is well defined since $\mathbf{\Phi}_{s,t}$ is non-decreasing.

With these notations in place $q_t^{n,n+1}$ can also be written as,

$$q_t^{n,n+1}((x,y),(x',y'))dx'dy = \frac{\prod_{i=1}^n \hat{m}(y_i')}{\prod_{i=1}^n \hat{m}(y_i)} \mathbb{P}\Big(\mathbf{\Phi}_{0,t}(x_i) \in dx_i', \mathbf{\Phi}_{0,t}^{-1}(y_j') \in dy_j \ \text{ for all } \ i,j\Big). \quad (1.22)$$

We now sketch an argument that gives the semigroup property $Q_{t+s}^{n,n+1} = Q_t^{n,n+1} Q_s^{n,n+1}$. We do not try to give all the details that would render it completely rigorous, mainly because it cannot be used to precisely describe the dynamics of $Q_t^{n,n+1}$, but nevertheless all the main steps are spelled out. We will however give a fully rigorous, elementary and self-contained treatment of such two-level couplings in the discrete setting arising from coalescing flows of birth and death chains in Section 5.2 of Chapter 5.

All equalities below should be understood after being integrated with respect to dx'' and dy over arbitrary Borel sets. The first equality is by definition. The second equality follows from the *cocycle* property and conditioning on the values of $\mathbf{\Phi}_{0,s}(x_i)$ and $\mathbf{\Phi}_{s,s+t}^{-1}(y_j'')$. Most importantly, this is where the *boundary behaviour assumptions* (1.6) and (1.7) we made at the beginning of this subsection are used. These ensure that no possible contributions from atoms on ∂I are missed; namely the random variable $\mathbf{\Phi}_{0,s}(x_i)$ is supported (its distribution gives full mass) in I°. Moreover, it is not too hard to see from the coalescing property of the flow that, we can restrict the integration over $(x',y') \in W^{n,n+1}(I^\circ)$ for otherwise the integrand vanishes. Finally, the third equality follows from *independence* of the increments and the fourth one by *stationarity* of the flow.

$$
\begin{aligned}
&q_{s+t}^{n,n+1}((x,y),(x'',y''))dx''dy = \frac{\prod_{i=1}^n \hat{m}(y_i'')}{\prod_{i=1}^n \hat{m}(y_i)}\mathbb{P}\Big(\mathbf{\Phi}_{0,s+t}(x_i)\in dx_i'', \mathbf{\Phi}_{0,s+t}^{-1}(y_j'')\in dy_j \text{ for all } i,j\Big)\\
&= \frac{\prod_{i=1}^n \hat{m}(y_i'')}{\prod_{i=1}^n \hat{m}(y_i)}\int_{(x',y')\in W^{n,n+1}(I^\circ)} \mathbb{P}\Big(\mathbf{\Phi}_{0,s}(x_i)\in dx_i', \mathbf{\Phi}_{s,s+t}(x_i')\in dx_i'', \mathbf{\Phi}_{0,s}^{-1}(y_j')\in dy_j, \mathbf{\Phi}_{s,s+t}^{-1}(y_j'')\in dy_j'\Big)\\
&= \int_{(x',y')\in W^{n,n+1}(I^\circ)} \frac{\prod_{i=1}^n \hat{m}(y_i')}{\prod_{i=1}^n \hat{m}(y_i)}\mathbb{P}\Big(\mathbf{\Phi}_{0,s}(x_i)\in dx_i', \mathbf{\Phi}_{0,s}^{-1}(y_j')\in dy_j\Big)\\
&\times \frac{\prod_{i=1}^n \hat{m}(y_i'')}{\prod_{i=1}^n \hat{m}(y_i')}\mathbb{P}\Big(\mathbf{\Phi}_{s,s+t}(x_i')\in dx_i'', \mathbf{\Phi}_{s,s+t}^{-1}(y_j'')\in dy_j'\Big)\\
&= \int_{(x',y')\in W^{n,n+1}(I^\circ)} q_s^{n,n+1}((x,y),(x',y'))q_t^{n,n+1}((x',y'),(x'',y''))dx'dy'dx''dy.
\end{aligned}
$$

1.2.4 Intertwinings and Markov functions

In this subsection (n_1,n_2) denotes one of $\{(n,n-1),(n,n),(n,n+1)\}$. First, recall the definitions of P_t^n and $\hat{P}_t^n$ given in (1.15) and (1.16) respectively. Similarly, we record here again, the following proposition and recall that it can in principle completely describe the evolution of the Y particles and characterizes the finite lifetime of the process Z as the killing time of Y.

Proposition 1.11. *For $t>0$ and f a bounded Borel function on $W^{n_1}(I^\circ)$ we have,*

$$
\Pi_{n_1,n_2}\hat{P}_t^{n_1}f = Q_t^{n_1,n_2}\Pi_{n_1,n_2}f. \tag{1.23}
$$

Now, we define the following integral operator Λ_{n_1,n_2} acting on Borel functions on $W^{n_1,n_2}(I^\circ)$, whenever f is integrable as,

$$
(\Lambda_{n_1,n_2}f)(x) = \int_{W^{n_1,n_2}(x)} \prod_{i=1}^{n_1}\hat{m}(y_i)f(x,y)dy,
$$

where we remind the reader that $\hat{m}(\cdot)$ is the density with respect to Lebesgue measure of the speed measure of the diffusion with generator $\hat{L}$.

The following intertwining relation is the fundamental ingredient needed for applying the theory of Markov functions, originating with the seminal paper of Rogers and Pitman [136]. This proposition as in the case of the one above directly follows by performing the dy integration in the explicit formula of the block determinant (or alternatively by invoking the coalescing property of the stochastic flow $(\mathbf{\Phi}_{s,t}(\cdot); s\le t)$ and the original definitions).

Proposition 1.12. *For $t>0$ we have the following equality of positive kernels,*

$$
P_t^{n_2}\Lambda_{n_1,n_2} = \Lambda_{n_1,n_2}Q_t^{n_1,n_2}. \tag{1.24}
$$

Combining the two propositions above gives the following relation for the Karlin-McGregor semigroups,

$$P_t^{n_2}\Lambda_{n_1,n_2}\Pi_{n_1,n_2} = \Lambda_{n_1,n_2}\Pi_{n_1,n_2}\hat{P}_t^{n_1}. \tag{1.25}$$

Namely, the two semigroups are themselves intertwined with kernel,

$$\left(\Lambda_{n_1,n_2}\Pi_{n_1,n_2}f\right)(x) = \int_{W^{n_1,n_2}(x)} \prod_{i=1}^{n_1} \hat{m}(y_1)f(y)dy.$$

This implies the following. Suppose $\hat{h}_{n_1}$ is a strictly positive (in $\mathring{W}^{n_1}$) eigenfunction for $\hat{P}_t^{n_1}$ namely, $\hat{P}_t^{n_1}\hat{h}_{n_1} = e^{\lambda_{n_1}t}\hat{h}_{n_1}$, then (with both sides possibly being infinite),

$$(P_t^{n_2}\Lambda_{n_1,n_2}\Pi_{n_1,n_2}\hat{h}_{n_1})(x) = e^{\lambda_{n_1}t}(\Lambda_{n_1,n_2}\Pi_{n_1,n_2}\hat{h}_{n_1})(x).$$

We are interested in strictly positive eigenfunctions because they allow us to define Markov processes, however non positive eigenfunctions can be built this way as well.

We now finally arrive at our main results. We need to make precise one more notion, already referenced several times in the introduction. For a possibly sub-Markov semigroup $(\mathfrak{P}_t; t \geq 0)$ or more generally, for fixed t, a sub-Markov kernel with eigenfunction $\mathfrak{h}$ with eigenvalue e^{ct} we define the Doob's h-transform by $e^{-ct}\mathfrak{h}^{-1} \circ \mathfrak{P}_t \circ \mathfrak{h}$. Observe that this is now an honest Markov semigroup (or Markov kernel).

If $\hat{h}_{n_1}$ is a strictly positive in $\mathring{W}^{n_1}$ eigenfunction for $\hat{P}_t^{n_1}$ then so it is for $Q_t^{n_1,n_2}$ from Proposition 1.11. We can thus define the proper Markov kernel $Q_t^{n_1,n_2,\hat{h}_{n_1}}$ which is the h-transform of $Q_t^{n_1,n_2}$ by $\hat{h}_{n_1}$. Define $h_{n_2}(x)$, strictly positive in $\mathring{W}^{n_2}$, as follows, assuming that the integrals are finite in the case of $W^{n,n}(I^\circ)$ and $W^{n+1,n}(I^\circ)$,

$$h_{n_2}(x) = (\Lambda_{n_1,n_2}\Pi_{n_1,n_2}\hat{h}_{n_1})(x),$$

and the Markov Kernel $\Lambda_{n_1,n_2}^{\hat{h}_{n_1}}(x,\cdot)$ with $x \in \mathring{W}^{n_2}$ by,

$$(\Lambda_{n_1,n_2}^{\hat{h}_{n_1}}f)(x) = \frac{1}{h_{n_2}(x)}\int_{W^{n_1,n_2}(x)} \prod_{i=1}^{n_1} \hat{m}(y_i)\hat{h}_{n_1}(y)f(x,y)dy.$$

Finally, defining $P_t^{n_2,h_{n_2}}$ to be the Karlin-McGregor semigroup $P_t^{n_2}$ h-transformed by h_{n_2} we obtain,

Proposition 1.13. *Let $Q_t^{n_1,n_2}$ denote one of the operators induced by the sub-Markov kernels on $W^{n_1,n_2}(I^\circ)$ defined in the previous subsection with the corresponding boundary conditions. Let $\hat{h}_{n_1}$ be a strictly positive eigenfunction for $\hat{P}_t^{n_1}$ and assume that $h_{n_2}(x) = (\Lambda_{n_1,n_2}\Pi_{n_1,n_2}\hat{h}_{n_1})(x)$ is finite in $W^{n_2}(I^\circ)$, so that in particular $\Lambda_{n_1,n_2}^{\hat{h}_{n_1}}$ is a Markov kernel. Then with the notations of the preceding* *paragraph we have the following relation for $t > 0$,*

$$P_t^{n_2,h_{n_2}} \Lambda_{n_1,n_2}^{\hat{h}_{n_1}} f = \Lambda_{n_1,n_2}^{\hat{h}_{n_1}} Q_t^{n_1,n_2,\hat{h}_{n_2}} f, \tag{1.26}$$

with f a bounded Borel function in $W^{n_1,n_2}(I^\circ)$.

This intertwining relation and the theory of Markov functions (see Section 2 of [136] for example) immediately imply the following corollary,

Corollary 1.14. *Assume $Z = (X, Y)$ is a Markov process with semigroup $Q_t^{n_1,n_2,\hat{h}_{n_2}}$, then the X component is distributed as a Markov process with semigroup $P_t^{n_2,h_{n_2}}$ started from x if (X, Y) is started from $\Lambda_{n_1,n_2}^{\hat{h}_{n_1}}(x, \cdot)$. Moreover, the conditional distribution of $Y(t)$ given $(X(s); s \le t)$ is $\Lambda_{n_1,n_2}^{\hat{h}_{n_1}}(X(t), \cdot)$.*

We give a final definition in the case of $W^{n,n+1}$ only, that has a natural analogue for $W^{n,n}$ and $W^{n+1,n}$ (we shall elaborate on the notion introduced below in Section 5.1 on well-posedness of *SDEs* with reflection). Take $Y = (Y_1, \cdots, Y_n)$ to be an n-dimensional system of *non-intersecting* paths in $\mathring{W}^n(I^\circ)$, so that in particular $Y_1 < Y_2 < \cdots < Y_n$. Then, by *X is a system of $n+1$ L-diffusions reflected off Y* we mean processes $(X_1(t), \cdots, X_{n+1}(t); t \ge 0)$, satisfying $X_1(t) \le Y_1(t) \le X_2(t) \le \cdots \le X_{n+1}(t)$ for all $t \ge 0$, and so that the following *SDEs* hold,

$$\begin{aligned}
dX_1(t) &= \sqrt{2a(X_1(t))}d\beta_1(t) + b(X_1(t))dt + dK^l(t) - dK_1^+(t),\\
&\vdots\\
dX_j(t) &= \sqrt{2a(X_j(t))}d\beta_j(t) + b(X_j(t))dt + dK_j^-(t) - dK_j^+(t),\\
&\vdots\\
dX_{n+1}(t) &= \sqrt{2a(X_{n+1}(t))}d\beta_{n+1}(t) + b(X_{n+1}(t))dt + dK_{n+1}^-(t) - dK^r(t).
\end{aligned} \tag{1.27}$$

where the positive finite variation processes K^l, K^r, K_i^+, K_i^- are such that K^l increases only when $X_1 = l$, K^r increases only when $X_{n+1} = r$, $K_i^+(t)$ increases only when $Y_i = X_i$ and $K_i^-(t)$ only when $Y_{i-1} = X_i$, so that $(X_1(t) \le Y_1(t) \le \cdots \le X_{n+1}(t)) \in W^{n,n+1}(I)$ forever. Here $\beta_1, \cdots, \beta_{n+1}$ are independent standard Brownian motions which are moreover *independent* of Y.

The reader should observe that the dynamics between (X, Y) are exactly the ones prescribed in the system of *SDEs* (1.14) with the difference being that now the process has infinite lifetime. This can be achieved from (1.14) by h-transforming the Y process as explained in this section to have infinite lifetime. By pathwise uniqueness of solutions to reflecting *SDEs* in continuous time-dependent domains, see Proposition 1.21 and also Section 5 of [6], under any absolutely continuous change of measure for the (X, Y)-process that depends only on Y (a Doob h-transform in particular), the equations (1.27) still hold with the β_i independent Brownian motions which moreover remain independent of the Y

process. We thus arrive at our main Theorem,

Theorem 1.15. *Suppose the assumptions of Proposition 1.5 and Proposition 1.13 hold and Y consists of n $\hat{L}$- diffusions h-transformed by $\hat{h}_n$ and X is a system of $n+1$ L-diffusions reflected off Y started from $\Lambda^{\hat{h}_n}_{n,n+1}(x,\cdot)$ with $x \in \mathring{W}^{n+1}(I)$. Then X is distributed as a diffusion process with semigroup $P_t^{n+1,h_{n+1}}$ started from x.*

The statement of the result for $W^{n,n}$ and $W^{n+1,n}$ is completely analogous.

Finally, the intertwining relation (1.26) also allows us to start the two-level process (X,Y) from a degenerate point, in particular the system of reflecting *SDEs* when some of the Y coordinates coincide, as long as starting the process with semigroup $P_t^{n_2,h_{n_2}}$ from such a degenerate point is valid. Suppose $\left(\mu_t^{n_2,h_{n_2}}\right)_{t>0}$ is an entrance law for $P_t^{n_2,h_{n_2}}$, namely for $t,s>0$,

$$\mu_s^{n_2,h_{n_2}} P_t^{n_2,h_{n_2}} = \mu_{t+s}^{n_2,h_{n_2}},$$

then we have the following corollary, which is obtained immediately by applying $\mu_t^{n_2,h_{n_2}}$ to both sides of (1.26):

Corollary 1.16. *Under the assumptions above, if $\left(\mu_s^{n_2,h_{n_2}}\right)_{s>0}$ is an entrance law for the process with semigroup $P_t^{n_2,h_{n_2}}$ then $\left(\mu_s^{n_2,h_{n_2}}\Lambda^{\hat{h}_{n_1}}_{n_1,n_2}\right)_{s>0}$ forms an entrance law for process (X,Y) with semigroup $Q_t^{n_1,n_2,\hat{h}_{n_1}}$.*

Hence, the statement of Theorem 1.15 generalizes, so that if X is a system of L-diffusions reflected off Y started according to an entrance law, then X is again itself distributed as a Markov process.

The entrance laws of interest in [6] correspond to starting the process with semigroup $P_t^{n_2,h_{n_2}}$ from a single point $(x,\cdots,x)$ for some $x \in I$. These are given by so called time dependent *biorthogonal ensembles*, namely measures of the form,

$$\det\left(f_i(t,x_j)\right)_{i,j=1}^{n_2} \det\left(g_i(t,x_j)\right)_{i,j=1}^{n_2} . \tag{1.28}$$

Under some further assumptions on the Taylor expansion of the one dimensional transition density $p_t(x,y)$ these are given by so called *polynomial ensembles*, where one of the determinant factors is the Vandermonde determinant,

$$\det\left(\phi_i(t,x_j)\right)_{i,j=1}^{n_2} \det\left(x_j^{i-1}\right)_{i,j=1}^{n_2} . \tag{1.29}$$

See the appendix of [6] for more details.

1.3 Consistent multilevel dynamics

1.3.1 General construction

Applying the theory developed in the previous section we will now construct consistent multilevel dynamics taking values in interlacing arrays. We will then give some simple examples related to the three classical unitarily invariant random matrix ensembles: the Gaussian (GUE), the Laguerre (LUE) and Jacobi (JUE). A construction related to the Hua-Pickrell measures is given in Section 3.6 of Chapter 3. For many more examples, in particular for a detailed study of Sturm-Liouville diffusions see Section 3 of [6].

First, recall that the space of continuous Gelfand-Tsetlin patterns of depth N denoted by $\mathbb{GT}_c(N)$ is defined to be,

$$\left\{\left(x^{(1)},\cdots,x^{(N)}\right): x^{(n)}\in W^n,\ x^{(n)}\prec x^{(n+1)}\right\},$$

and also the space of continuous symplectic Gelfand-Tsetlin patterns of depth N denoted by $\mathbb{GT}_{c,\mathbf{s}}(N)$ is given by,

$$\left\{\left(x^{(1)},\hat{x}^{(1)}\cdots,x^{(N)},\hat{x}^{(N)}\right): x^{(n)},\hat{x}^{(n)}\in W^n,\ x^{(n)}\prec\hat{x}^{(n)}\prec x^{(n+1)}\right\}.$$

We will describe the construction for $\mathbb{GT}_c$, with the extension to $\mathbb{GT}_{c,\mathbf{s}}$ being analogous. Let us fix an interval I with endpoints $l<r$ and let L_n for $n=1,\cdots,N$ be a sequence of diffusion process generators in I (satisfying (1.6) and (1.7)) given by,

$$L_n=a_n(x)\frac{d^2}{dx^2}+b_n(x)\frac{d}{dx}. \tag{1.30}$$

We will moreover denote their transition densities with respect to Lebesgue measure by $p_t^n(\cdot,\cdot)$.

We want to consider a process $(\mathbb{X}(t);t\geq 0)=\left(\left(\mathbb{X}^{(1)}(t),\cdots,\mathbb{X}^{(N)}(t)\right);t\geq 0\right)$ taking values in $\mathbb{GT}_c(N)$ so that, for each $2\leq n\leq N$, $\mathbb{X}^{(n)}$ consists of n independent L_n diffusions reflected off the paths of $\mathbb{X}^{(n-1)}$. More precisely we consider the following system of reflecting *SDEs*, with $1\leq i\leq n\leq N$, initialized in $\mathbb{GT}_c(N)$ and stopped at the stopping time $\tau_{\mathbb{GT}_c(N)}$ to be defined below,

$$d\mathbb{X}_i^{(n)}(t)=\sqrt{2a_n\left(\mathbb{X}_i^{(n)}(t)\right)}d\beta_i^{(n)}(t)+b_n\left(\mathbb{X}_i^{(n)}(t)\right)dt+dK_i^{(n),-}-dK_i^{(n),+}, \tag{1.31}$$

driven by an array $\left(\beta_i^{(n)}(t);t\geq 0,1\leq i\leq n\leq N\right)$ of $\frac{N(N+1)}{2}$ independent standard Brownian motions. The positive finite variation processes $K_i^{(n),-}$ and $K_i^{(n),+}$ are such that $K_i^{(n),-}$ increases only when $\mathbb{X}_i^{(n)}=\mathbb{X}_{i-1}^{(n-1)}$, $K_i^{(n),+}$ increases only when $\mathbb{X}_i^{(n)}=\mathbb{X}_i^{(n-1)}$ with $K_1^{(N),-}$ increasing when $\mathbb{X}_1^{(N)}=l$ and $K_N^{(N),+}$ increasing when $\mathbb{X}_N^{(N)}=r$, so that $\mathbb{X}=\left(\mathbb{X}^{(1)},\cdots,\mathbb{X}^{(N)}\right)$ stays in $\mathbb{GT}_c(N)$

forever. The stopping time $\tau_{\mathbb{GT}_c(N)}$ is given by,

$$\tau_{\mathbb{GT}_c(N)} = \inf\left\{t \geq 0 : \exists\, (n,i,j)\; 2 \leq n \leq N-1, 1 \leq i < j \leq n \text{ s.t. } \mathbb{X}_i^{(n)}(t) = \mathbb{X}_j^{(n)}(t)\right\}.$$

Stopping at $\tau_{\mathbb{GT}_c(N)}$ takes care of the problematic possibility of two of the time dependent barriers coming together. It will turn out that $\tau_{\mathbb{GT}_c(N)} = \infty$ almost surely under certain initial conditions of interest to us given in Proposition 1.17 below; this will be the case since then each level $\mathbb{X}^{(n)}$ will evolve according to a Doob's h-transform and thus consisting of non-intersecting paths. The system of reflecting *SDEs* (1.31) above is well-posed, under a Yamada-Watanabe condition on the coefficients $\left(\sqrt{a_n}, b_n\right)$ for $1 \leq n \leq N$ (see Proposition 1.21).

We would like Theorem 1.15 to be applicable to each pair $(\mathbb{X}^{(n-1)}, \mathbb{X}^{(n)})$. To this end, for $n = 2, \cdots, N$, suppose that $\mathbb{X}^{(n-1)}$ is distributed according to the following h-transformed Karlin-McGregor semigroup by the strictly positive in $\mathring{W}^{n-1}$ eigenfunction g_{n-1} with eigenvalue $e^{c_{n-1}t}$,

$$e^{-c_{n-1}t} \frac{g_{n-1}(y_1, \cdots, y_{n-1})}{g_{n-1}(x_1, \cdots, x_{n-1})} \det\left(\widehat{p}_t^n(x_i, y_j)\right)_{i,j=1}^{n-1},$$

where $\widehat{p}_t^n(\cdot, \cdot)$ denotes the transition density associated with the dual $\widehat{L_n}$ (killed at an exit of regular absorbing boundary point) of L_n. We furthermore, denote by $\widehat{m^n}(\cdot)$ the density with respect to Lebesgue measure of the speed measure of $\widehat{L_n}$. Then, Theorem 1.15 gives that under a special initial condition (stated therein) for the joint dynamics of $(\mathbb{X}^{(n-1)}, \mathbb{X}^{(n)})$ the projection on $\mathbb{X}^{(n)}$ is distributed as the G_{n-1} h-transform of n independent L_n diffusions, thus consisting of non-intersecting paths, where G_{n-1} is given by,

$$G_{n-1}(x_1, \cdots, x_n) = \int_{W^{n-1,n}(x)} \prod_{i=1}^{n-1} \widehat{m^n}(y_i) g_{n-1}(y_1, \cdots, y_{n-1}) dy_1 \cdots dy_{n-1}. \tag{1.32}$$

Consistency then demands, by comparing $(\mathbb{X}^{(n-1)}, \mathbb{X}^{(n)})$ and $(\mathbb{X}^{(n)}, \mathbb{X}^{(n+1)})$, the following condition between the transition kernels (which is also sufficient as we see below for the construction of a consistent process $(\mathbb{X}^{(1)}, \cdots, \mathbb{X}^{(N)})$), for $t > 0, x, y \in \mathring{W}^n$,

$$e^{-c_{n-1}t} \frac{G_{n-1}(y_1, \cdots, y_n)}{G_{n-1}(x_1, \cdots, x_n)} \det\left(p_t^n(x_i, y_j)\right)_{i,j=1}^{n} = e^{-c_n t} \frac{g_n(y_1, \cdots, y_n)}{g_n(x_1, \cdots, x_n)} \det\left(\widehat{p}_t^{n+1}(x_i, y_j)\right)_{i,j=1}^{n}. \tag{1.33}$$

Denote the semigroup associated with these densities by $\left(\mathfrak{P}^{(n)}(t); t > 0\right)$ and also define the Markov kernels $\mathfrak{L}_{n-1}^n(x, dy)$ for $x \in \mathring{W}^n$ by,

$$\mathfrak{L}_{n-1}^n(x, dy) = \frac{\prod_{i=1}^{n-1} \widehat{m^n}(y_i) g_{n-1}(y_1, \cdots, y_{n-1})}{G_{n-1}(x_1, \cdots, x_n)} \mathbf{1}\left(y \in W^{n-1,n}(x)\right) dy_1 \cdots dy_{n-1}.$$

Then, by inductively applying Theorem 1.15, we see that the following Proposition holds:

Proposition 1.17. *Under the assumptions of Theorem 1.15, we moreover suppose that for* $2 \leq n \leq N-1$*, the consistency relations (1.32) and (1.33) hold. Let* $\nu_N(dx)$ *be a measure supported in* $\mathring{W}^N$*. Consider the process* $(\mathbb{X}(t); t \geq 0) = \left(\left(\mathbb{X}^{(1)}(t), \cdots, \mathbb{X}^{(N)}(t)\right); t \geq 0\right)$ *in* $\mathbb{GT}_c(N)$ *satisfying the SDEs (1.31) and initialized according to,*

$$\nu_N(dx^{(N)})\mathfrak{L}^N_{N-1}(x^{(N)}, dx^{(N-1)}) \cdots \mathfrak{L}^2_1(x^{(2)}, dx^{(1)}). \tag{1.34}$$

Then $\tau_{\mathbb{GT}_c(N)} = \infty$ *almost surely,* $\left(\mathbb{X}^{(n)}(t); t \geq 0\right)$ *for* $1 \leq n \leq N$ *evolves according to* $\mathfrak{P}^{(n)}(t)$ *and for fixed* $T > 0$ *the law of* $\left(\mathbb{X}^{(1)}(T), \cdots, \mathbb{X}^{(N)}(T)\right)$ *is given by,*

$$\left(\nu_N \mathfrak{P}^{(N)}_T\right)(dx^{(N)})\mathfrak{L}^N_{N-1}(x^{(N)}, dx^{(N-1)}) \cdots \mathfrak{L}^2_1(x^{(2)}, dx^{(1)}). \tag{1.35}$$

Proof. For $n = 2$ this is the statement of Theorem 1.15. Assume that the proposition is proven for $n = N-1$. Observe that, an initial condition of the form (1.34) in $\mathbb{GT}_c(N)$ gives rise to an initial condition of the same form in $\mathbb{GT}_c(N-1)$:

$$\begin{aligned}
&\tilde{\nu}_{N-1}(dx^{(N-1)})\mathfrak{L}^{N-1}_{N-2}(x^{(N-1)}, dx^{(N-2)}) \cdots \mathfrak{L}^2_1(x^{(2)}, dx^{(1)}),\\
&\tilde{\nu}_{N-1}(dx^{(N-1)}) = \int_{\mathring{W}^N} \nu_N(dx^{(N)})\mathfrak{L}^N_{N-1}(x^{(N)}, dx^{(N-1)}).
\end{aligned}$$

Then, by the inductive hypothesis $\left(\mathbb{X}^{(N-1)}(t); t \geq 0\right)$ evolves according to $\mathfrak{P}^{(N-1)}(t)$, with the joint evolution of $(\mathbb{X}^{(N-1)}, \mathbb{X}^{(N)})$, by (1.32) and (1.33) with $n = N-1$, as in Theorem 1.15 and with initial condition $\nu_N(dx^{(N)})\mathfrak{L}^N_{N-1}(x^{(N)}, dx^{(N-1)})$. We thus obtain that $\left(\mathbb{X}^{(N)}(t); t \geq 0\right)$ evolves according to $\mathfrak{P}^{(N)}(t)$ and for fixed T the conditional distribution of $\mathbb{X}^{(N-1)}(T)$ given $\mathbb{X}^{(N)}(T)$ is $\mathfrak{L}^N_{N-1}\left(\mathbb{X}^{(N)}(T), dx^{(N-1)}\right)$. This, along with the inductive hypothesis on the law of $\mathbb{GT}_c(N-1)$ at time T yields (1.35). The fact that $\tau_{\mathbb{GT}_c(N)} = \infty$ is also clear since each $\left(\mathbb{X}^{(n)}(t); t \geq 0\right)$ is governed by a Doob transformed Karlin-McGregor semigroup. □

1.3.2 Examples

We will now give some examples of such constructions in $\mathbb{GT}_c(N)$. In all cases the Markov kernels $\mathfrak{L}^n_{n-1}$ are given by the 'Vandermonde links':

$$\mathfrak{L}^n_{n-1}(x, dy) = \frac{(n-1)!\Delta_{n-1}(y)}{\Delta_n(x)} \mathbf{1}\left(y \prec x\right) dy_1 \cdots dy_{n-1}, \tag{1.36}$$

where $\Delta_n(x) = \prod_{1 \leq i < j \leq n}(x_j - x_i)$ is the Vandermonde determinant. Moreover, the semigroups $\mathfrak{P}^{(n)}(t)$ have transition kernels of the form:

$$e^{-\lambda_n t}\frac{\Delta_n(y)}{\Delta_n(x)} \det\left(p^n_t\left(x_i, y_j\right)\right)^n_{i,j=1}. \tag{1.37}$$

Brownian motion

$$a_n(x) = \frac{1}{2}\ ,\ b_n(x) = 0\ ,\ \lambda_n = 0.$$

This is the original construction of Warren. The distribution at time T of $\left(\mathbb{X}^{(1)}(T), \cdots, \mathbb{X}^{(N)}(T)\right)$ if started from the origin is given by the GUE minor/corners process with diffusivity T, see [164].

Ornstein-Uhlenbeck

$$a_n(x) = \frac{1}{2}\ ,\ b_n(x) = -x\ ,\ \lambda_n = -\frac{n(n+1)}{2}.$$

This is the stationary analogue of the Brownian motion model. It leaves the GUE minor process invariant. For more details see Section 3.8 of [6].

The following three constructions related to the LUE and JUE ensembles are new (see also [149]):

Squared Bessel process

$$a_n(x) = 2x\ ,\ b_n(x) = d + 2(N-n)\ ,\ \lambda_n = 0.$$

The distribution of this diffusion at time T if started from the origin is given by the LUE minor process (with certain parameters, for more details see Section 3.7 of [6]).

Laguerre diffusion

$$a_n(x) = 2x\ ,\ b_n(x) = d + 2(N-n) - 2x\ ,\ \lambda_n = -n(n+1).$$

This diffusion in $\mathbb{GT}_c(N)$ leaves the LUE minor process invariant, for more details see Section 3.8 of [6].

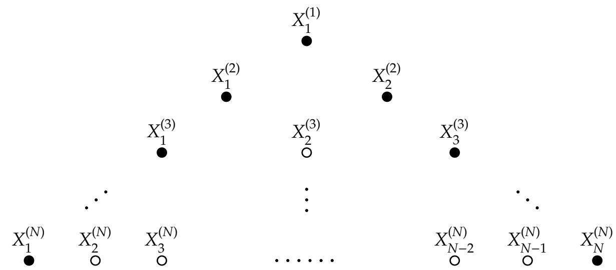
Jacobi diffusion

$$\begin{aligned} a_n(x) = 2x(1-x)\ ,\ b_n(x) = 2\left(\alpha_1 + (N-n)\right) - 2\left(2(N-n) + (\alpha_1+\alpha_2)\right)x, \\ \lambda_n = -n(n+1)\left[\frac{2(n-1)}{3} + 2(N-n) + (\alpha_1+\alpha_2)\right]. \end{aligned}$$

This diffusion in $\mathbb{GT}_c(N)$ leaves the JUE minor process invariant, for more details see Section 3.8 of [6]

1.4 Edge particle systems

In this section we will study the autonomous particle systems at either edge of the Gelfand-Tsetlin pattern valued processes we have constructed. In the figure below, the particles we will be concerned with are denoted in $\bullet$.



Our goal is to derive determinantal expressions for their transition densities. Such expressions were derived by Schutz for TASEP in [141] and later Warren [164] for Brownian motions. See also Johansson's work in [88], for an analogous formula for a Markov chain related to the Meixner ensemble and finally Dieker and Warren's investigation in [58], for formulae in the discrete setting based on the RSK correspondence. These so called Schutz-type formulae were the starting points for the recent complete solution of TASEP in [105] which led to the KPZ fixed point and also for the recent progress [89] in the study of the two time joint distribution in Brownian directed percolation. For a detailed investigation of the Brownian motion model the reader is referred to the book [167].

We will mainly restrict ourselves to the consideration of Brownian motions, $BESQ(d)$ processes and the diffusions associated with orthogonal polynomials. In a little bit more generality we will assume that the interacting diffusions have generators,

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx},$$

with,

$$a(x) = a_0 + a_1 x + a_2 x^2 \ , \ b(x) = b_0 + b_1 x.$$

We will also make the following **standing assumption** in this section. We restrict to the case of the boundaries of the state space I being either *natural* or *entrance* thus the state space is an open interval (l, r). Under these assumptions the transition densities will be smooth in (l, r) in both the backwards and forwards variables (possibly blowing up as we approach l or r see e.g [147]). This covers all the processes we built in the subsection above that are related to random matrix theory. This interacting particle system can also be seen as the solution to the following system of SDE's with one-sided collisions with $(x_1^1 \leq \cdots \leq x_N^N)$,

$$
\begin{aligned}
X_1^{(1)}(t) &= x_1^1 + \int_0^t \sqrt{2a(X_1^{(1)}(s))}d\gamma_1^1(s) + \int_0^t b^{(1)}(X_1^{(1)}(s))ds, \\
&\vdots \\
X_m^{(m)}(t) &= x_m^m + \int_0^t \sqrt{2a(X_m^{(m)}(s))}d\gamma_m^m(s) + \int_0^t b^{(m)}(X_m^{(m)}(s))ds + K_m^{m,-}(t), \qquad (1.38)\\
&\vdots \\
X_N^{(N)}(t) &= x_N^N + \int_0^t \sqrt{2a(X_N^{(N)}(s))}d\gamma_N^N(s) + \int_0^t b^{(N)}(X_N^{(N)}(s))ds + K_N^{N,-}(t).
\end{aligned}
$$

where γ_i^i are independent standard Brownian motions and $K_i^{i,-}$ are positive finite variation processes with the measure $dK_i^{i,-}$ supported on $\left\{t : X_i^{(i)}(t) = X_{i-1}^{(i-1)}(t)\right\}$ and

$$b^{(k)}(x) = b(x) + (N-k)a'(x) = b_0 + (N-k)a_1 + (b_1 + 2(N-k)a_2)x.$$

These $SDEs$ are well posed under a Yamada-Watanabe condition, see Proposition 1.21. See the following figure for a description of the interaction. The arrows indicate the direction of the 'pushing force' (with magnitude the finite variation process K) applied when collisions occur between the particles so that the ordering is maintained.

$$\overset{X_1^{(1)}}{\bullet} \longrightarrow \overset{X_2^{(2)}}{\bullet} \longrightarrow \overset{X_3^{(3)}}{\bullet} \quad \cdots \quad \overset{X_{N-1}^{(N-1)}}{\bullet} \longrightarrow \overset{X_N^{(N)}}{\bullet}.$$

Note that our assumption that the boundary points are either *entrance* or *natural* does not always allow for an *infinite* such particle system, in particular think of the squared Bessel BESQ(d) case described in Subsection 1.3.2 where d drops down by 2 each time we add a particle. Denote by $p_t^{(k)}(x,y)$ the transition kernel associated with the $L^{(k)}$-diffusion with generator,

$$L^{(k)} = a(x)\frac{d^2}{dx^2} + b^{(k)}(x)\frac{d}{dx}.$$

Define the constant $c_{k,N} = 2(N-k-1)a_2 + b_1$ and note that the $L^{(k)}$-diffusion is the h transform of the conjugate $\widehat{L^{(k+1)}}$ with $\widehat{m^{(k+1)}}^{-1}(x)$ with eigenvalue $c_{k,N}$, so that $L^{(k)}$ is exactly $\left(\widehat{L^{(k+1)}}\right)^* - c_{k,N}$ which is again a bona fide diffusion process generator (with L^* denoting the formal adjoint of L with respect to Lebesgue measure). Now, making use of (1.4) and (1.5) we obtain the following relation between the transition densities,

$$
\begin{aligned}
p_t^{(k)}(x,z) &= -e^{c_{k,N}t}\int_l^z \partial_x p_t^{(k+1)}(x,w)dw, \qquad (1.39)\\
\partial_z^j p_t^{(k)}(x,z) &= -e^{c_{k,N}t}\partial_z^{j-1}\partial_x p_t^{(k+1)}(x,z).
\end{aligned}
$$

Defining,

$$\mathcal{S}_t^{(k),j}(x,x') = \begin{cases} \int_l^{x'} \frac{(x'-z)^{j-1}}{(j-1)!} p_t^{(k)}(x,z)dz, & j \geq 1 \\ \partial_{x'}^{-j} p_t^{(k)}(x,x'), & j \leq 0 \end{cases},$$

and with $x = (x_1, \cdots, x_N)$, $x' = (x_1', \cdots, x_N')$,

$$s_t(x,x') = \det\left(\mathcal{S}_t^{(i),i-j}(x_i, x_j')\right)_{i,j\leq N}, \tag{1.40}$$

we arrive at the following proposition, see Subsection 1.5.3 for a sketch of a proof.

Proposition 1.18. *The process $(X_1^{(1)}(t), \cdots, X_N^{(N)}(t))$ has transition densities $s_t(x,x')$.*

In the standard Brownian motion case with $p_t^{(k)}$ the heat kernel this recovers Proposition 8 from [164].

Now, we consider the interacting particle system at the other edge of the pattern with the i^{th} particle getting reflected downwards from the $i-1^{th}$, namely with $(x_1^1 \geq \cdots \geq x_1^N)$ this is given by the following system of *SDEs* with reflection,

$$\begin{aligned} X_1^{(1)}(t) &= x_1^1 + \int_0^t \sqrt{2a(X_1^{(1)}(s))}d\gamma_1^1(s) + \int_0^t b^{(1)}(X_1^{(1)}(s))ds, \\ &\vdots \\ X_1^{(m)}(t) &= x_1^m + \int_0^t \sqrt{2a(X_1^{(m)}(s))}d\gamma_1^m(s) + \int_0^t b^{(m)}(X_1^{(m)}(s))ds - K_1^{m,+}(t), \\ &\vdots \\ X_1^{(N)}(t) &= x_1^N + \int_0^t \sqrt{2a(X_1^{(N)}(s))}d\gamma_1^N(s) + \int_0^t b^{(N)}(X_1^{(N)}(s))ds - K_1^{N,+}(t), \end{aligned} \tag{1.41}$$

where γ_1^i are independent standard Brownian motions and $K_1^{i,+}$ are positive finite variation processes with the measure $dK_1^{i,+}$ supported on $\left\{t : X_i^{(i)}(t) = X_{i-1}^{(i-1)}(t)\right\}$. Again see the figure below,

$$\overset{X_1^{(N)}}{\bullet} \longleftarrow \overset{X_1^{(N-1)}}{\bullet} \longleftarrow \overset{X_1^{(N-2)}}{\bullet} \quad \cdots \quad \overset{X_1^{(2)}}{\bullet} \longleftarrow \overset{X_1^{(1)}}{\bullet}.$$

Note that we also have the following relation for the transition kernel of the k^{th} particle, which is immediate from (1.39) since each diffusion process in this section is an honest Markov process, so that,

$$p_t^{(k)}(x,z) = e^{c_{k,N}t} \int_z^r \partial_x p_t^{(k+1)}(x,w)dw.$$

Define,

$$\bar{\mathcal{S}}_t^{(k),j}(x,x') = \begin{cases} -\int_{x'}^{r} \frac{(x'-z)^{j-1}}{(j-1)!} p_t^{(k)}(x,z)dz, & j \geq 1 \\ \partial_{x'}^{-j} p_t^{(k)}(x,x'), & j \leq 0 \end{cases}.$$

Then letting, with $x = (x_1, \cdots, x_N)$, $x' = (x'_1, \cdots, x'_N)$,

$$\bar{s}_t(x,x') = \det(\bar{\mathcal{S}}_t^{(i),i-j}(x_i, x'_j))_{i,j\leq N}, \tag{1.42}$$

we arrive at the following proposition.

Proposition 1.19. *The process* $(X_1^{(1)}(t), \cdots, X_1^{(N)}(t))$ *has transition densities* $\bar{s}_t(x,x')$.

Via a simple integration, we obtain the following formulae for the distributions of the leftmost and rightmost particles in the Gelfand-Tsetlin pattern,

Corollary 1.20.

$$\mathbb{P}_{x^{(0)}}(X_N^{(N)}(t) \leq z) = \det\left(\mathcal{S}_t^{(i),i-j+1}(x_i^{(0)}, z)\right)_{i,j=1}^N,$$
$$\mathbb{P}_{\bar{x}^{(0)}}(X_1^{(N)}(t) \geq z) = \det\left(-\bar{\mathcal{S}}_t^{(i),i-j+1}(\bar{x}_i^{(0)}, z)\right)_{i,j=1}^N,$$

where $x^{(0)} = (x_1^{(0)} \leq \cdots \leq x_N^{(0)})$ *and* $\bar{x}^{(0)} = (\bar{x}_1^{(0)} \geq \cdots \geq \bar{x}_N^{(0)})$.

For $p_t^{(k)}$ the heat kernel and $x^{(0)} = (0, \cdots, 0)$ this recovers a formula from [164]. In the *BESQ*(d) case and $t = 1$ the above give expressions for the largest and smallest eigenvalues for the *LUE* ensemble. We obtain the analogous expressions in the Jacobi case as $t \to \infty$ since the *JUE* is the invariant measure of non-intersecting Jacobi processes.

1.5 On the proofs of Propositions 1.5, 1.9 1.18, 1.19

To rigorously establish Propositions 1.5, 1.9, 1.18, 1.19 one proceeds in two steps. First, we need well-posedness of the *SDEs* with reflection and then we show that the transition densities of these Markovian evolutions are given by the block determinant kernels $q_t^{n_1,n_2}$. We will sketch the strategy of proof, also giving some of the key ingredients (for the details the reader is referred to Section 5 of [6]).

1.5.1 Well-posedness of SDEs with reflection

Regarding well-posedness we introduce the following condition that we abbreviate **YW**, after Yamada-Watanabe. Let I be an interval with endpoints $l < r$ and suppose ρ is a Borel function from $(0,\infty)$ to itself such that $\int_{0^+} \frac{dx}{\rho(x)} = \infty$. Assume the functions $a : I \to \mathbb{R}_+$ and $b : I \to \mathbb{R}$ satisfy (where we implicitly assume that a and b initially defined in I° can be extended continuously to the boundary points l and r):

$$
\begin{aligned}
|\sqrt{a(x)}-\sqrt{a(y)}|^2 &\le \rho(|x-y|),\\
|b(x)-b(y)| &\le C|x-y|.
\end{aligned}
$$

Then, we have the following result:

Proposition 1.21. *Under the* **YW** *assumption on* $(\sqrt{2a}, b)$*, the systems of SDEs with reflection (1.14), (1.20), (1.31), (1.38) and (1.41) (in case of 1.31 the assumption is enforced for all n) have a pathwise unique strong, in particular Markovian, solution.*

The rigorous construction of *SDEs* with reflection goes through the so called Skorokhod problem and its solution map $\mathcal{S}_{\mathrm{sol}}$ (see for example [145] and Section 5 of [6] for an exposition).

The proof of Proposition 1.21 is then split into two steps. First, one shows weak existence of solutions to reflecting *SDEs* with merely continuous coefficients with at most linear growth. The essential tools are the Lipschitz continuity of the Skorokhod solution map $\mathcal{S}_{\mathrm{sol}}$ (see [145], [44]), the Skorokhod Representation Theorem and the Martingale Representation Theorem (see for example Chapter 5 of [134]).

Then, because of the intrinsic one-dimensionality of the problem, one can obtain pathwise uniqueness under assumption **YW** by extending a classical argument of Le Gall (see Chapter 9 of [134]) to *SDEs* with reflection.

Finally, observe that Proposition 1.21 covers all examples given in Subsection 1.3.2.

1.5.2 On the proofs of Propositions 1.5, 1.9

We now move on and introduce the following assumption:

Standing assumption We enforce the assumptions of Sections 1.2.1 and 1.2.2 on the L-diffusion, namely that $a(x) \in C^1(I^\circ)$ with $a(x) > 0, \forall x \in I^\circ$ and $b(x) \in C(I^\circ)$ and depending on which of $W^{n,n+1}, W^{n,n}$ or $W^{n+1,n}$ our processes take values in the corresponding pair of boundary conditions ((1.6),(1.7)),((1.8),(1.9)) or ((1.10),(1.11)) at l and r. Moreover, we assume the **YW** condition so that the systems of *SDEs* are well posed.

Under this standing assumption we have the following results (cf. Propositions 1.5, 1.9):

Proposition 1.22. *Assume l and r are either natural or entrance for the L-diffusion. Then $q_t^{n,n+1}$ form the transition densities for the system of SDEs (1.14).*

Proposition 1.23. *Assume l is either natural or exit and r is either natural or entrance for the L-diffusion. Then $q_t^{n,n}$ form the transition densities for the system of SDEs (1.20).*

Proposition 1.24. *Assume l and r are regular reflecting for the L-diffusion and $\lim_{x\to l,r} a(x) > 0$ and $\lim_{x\to l,r} b(x)$, $\lim_{x\to l,r}(a'(x)-b(x))$ exist and are finite. Then $q_t^{n,n+1}$ form the transition densities for the system of SDEs (1.14).*

Proposition 1.24 has an exact analogue for $q_t^{n,n}$ and the system of *SDEs* (1.20) which we omit.

The proof of Propositions 1.22, 1.23, 1.24 makes use of Ito's formula and essentially checks Kolmogorov's backward equation, along with reflecting/Neumann boundary conditions. For example in the case of Proposition 1.22, for fixed $(x', y') \in \mathring{W}^{n,n+1}(I^\circ)$:

$$\partial_t q_t^{n,n+1}((x,y),(x',y')) = \left(\sum_{i=1}^{n+1} \mathcal{D}_m^{x_i}\mathcal{D}_s^{x_i} + \sum_{i=i}^{n} \mathcal{D}_{\hat{m}}^{y_i}\mathcal{D}_{\hat{s}}^{y_i}\right) q_t^{n,n+1}((x,y),(x',y')),$$
$$\text{in } (0,\infty) \times \mathring{W}^{n,n+1}(I^\circ),$$

with,

$$\partial_{x_i} q_t^{n,n+1}((x,y),(x',y'))|_{x_i=y_i} = 0 \ , \ \partial_{x_i} q_t^{n,n+1}((x,y),(x',y'))|_{x_i=y_{i-1}} = 0.$$

The differential equation is a consequence of the multilinearity of determinants along with the following properties of the entries (where we slightly abuse notation and use the same notation for both the scalar entries and the matrices that come into the definition of $q_t^{n,n+1}$): for $x, y \in I^\circ$,

$$\partial_t A_t(x,x') = \mathcal{D}_m^x \mathcal{D}_s^x A_t(x,x') \ \ , \ \ \partial_t B_t(x,y') = \mathcal{D}_m^x \mathcal{D}_s^x B_t(x,y'),$$
$$\partial_t C_t(y,x') = \mathcal{D}_{\hat{m}}^y \mathcal{D}_{\hat{s}}^y C_t(y,x') \ \ , \ \ \partial_t D_t(y,y') = \mathcal{D}_{\hat{m}}^y \mathcal{D}_{\hat{s}}^y D_t(y,y').$$

To see the equation for $C_t(y,x')$ note that since $\mathcal{D}_{\hat{m}} = \mathcal{D}_s$ and $\mathcal{D}_{\hat{s}} = \mathcal{D}_m$ we have,

$$\partial_t C_t(y,x') = -\mathcal{D}_s^y \partial_t p_t(y,x') = -\mathcal{D}_s^y \mathcal{D}_m^y \mathcal{D}_s^y p_t(y,x') = -\mathcal{D}_{\hat{m}}^y \mathcal{D}_{\hat{s}}^y \mathcal{D}_s^y p_t(y,x') = \mathcal{D}_{\hat{m}}^y \mathcal{D}_{\hat{s}}^y C_t(y,x').$$

While, the Neumann boundary conditions are a consequence of, with $x, y \in I^\circ$,

$$\partial_x A_t(x,x')|_{x=y} = -\hat{m}(y) C_t(y,x'),$$
$$\partial_x B_t(x,y')|_{x=y} = -\hat{m}(y) D_t(y,y').$$

The extra conditions in Proposition 1.24 when the boundary points l and r are accessible are so that we can apply Ito's formula (which requires a C^2 function in an open domain).

1.5.3 On the proofs of Propositions 1.18, 1.19

Finally, the proofs of Propositions 1.18, 1.19 also follow the same strategy. Again we apply Ito's formula and check the Kolmogorov backwards equation. For example regarding Proposition 1.18, with the notations of Section 1.4, we have the differential equation in

$(0,\infty) \times \mathring{W}^N(I) \times \mathring{W}^N(I)$:

$$\partial_t s_t(x,x') = \sum_{i=1}^{N} L_{x_i}^{(k)} s_t(x,x'),$$

with Neumann/reflecting boundary conditions:

$$\partial_{x_i} s_t(x,x')|_{x_i=x_{i-1}} = 0 \text{ for } i = 2,\cdots,N.$$

The differential equation is satisfied since we have $\partial_t \mathcal{S}_t^{(k),j}(x,x') = L_x^{(k)} \mathcal{S}_t^{(k),j}(x,x')$ for all k. Moreover, the Neumann boundary conditions follow since:

$$\partial_{x_i} \mathcal{S}_t^{(i),i-j}(x_i,x'_j)|_{x_i=x_{i-1}} = -e^{-c_{i-1,N}t} \mathcal{S}_t^{(i-1),i-1-j}(x_{i-1},x'_j).$$

This is true because of the following observations. For $j \leq -1$:

$$\partial_z^{-j} p_t^{(i-1)}(x,z) = -e^{c_{i-1,N}t} \partial_z^{-j-1} \partial_x p_t^{(i)}(x,z).$$

While, for $j \geq 1$:

$$\begin{aligned}
&\int_l^{x'} \frac{(x'-z)^{j-1}}{(j-1)!} p_t^{(i-1)}(x,z)dz = -e^{c_{i-1,N}t} \partial_x \int_l^{x'} \frac{(x'-z)^{j-1}}{(j-1)!} \int_l^z p_t^{(i)}(x,w)dwdz \\
&= -e^{c_{i-1,N}t} \partial_x \Big[\Big[-\frac{(x'-z)^j}{j!} \int_l^z p_t^{(k)}(x,w)dw \Big]_l^{x'} - \int_l^{x'} -\frac{(x'-z)^j}{j!} p_t^{(i)}(x,z)dz \Big] \\
&= -e^{c_{i-1,N}t} \partial_x \int_l^{x'} \frac{(x'-z)^j}{j!} p_t^{(i)}(x,z)dz.
\end{aligned}$$

And thus, $\mathcal{S}_t^{(i-1),j}(x,x') = -e^{c_{i-1,N}t} \partial_x \mathcal{S}_t^{(i),j+1}(x,x')$.

Chapter 2

Consistent dynamics for β-ensembles

2.1 Introduction

The aim of this short chapter is to establish intertwining relations between the semigroups of general β-Laguerre and β-Jacobi processes, in analogy to the ones obtained for general β-Dyson Brownian motion in [131] (see also [75]). These, also generalize the relations obtained for $\beta = 2$ in chapter 1 when the transition kernels for these semigroups are given explicitly in terms of h-transforms of Karlin-McGregor determinants.

We begin, by introducing the stochastic processes we will be dealing with. Consider the unique strong solution to the following system of *SDEs* with $i = 1, \cdots, n$ with values in $[0, \infty)^n$,

$$dX_i^{(n)}(t) = 2\sqrt{X_i^{(n)}(t)}dB_i^{(n)}(t) + \beta\left(\frac{d}{2} + \sum_{1\le j\le n, j\ne i}\frac{2X_i^{(n)}(t)}{X_i^{(n)}(t) - X_j^{(n)}(t)}\right)dt, \tag{2.1}$$

where the $B_i^{(n)}$, $i = 1, \cdots, n$, are independent standard Brownian motions. This process, was introduced and studied by Demni in [55] in relation to Dunkl processes, (see for example [138]) where it is referred to as the β-Laguerre process, since its distribution at time 1, if started from the origin, is given by the β-Laguerre ensemble (see Section 5 of [55]). We could, equally well, have called this the β-squared Bessel process, since for $\beta = 2$ it exactly consists of n $BESQ(d)$ diffusion processes conditioned to never collide as first proven in [110] but we stick to the terminology of [55]. Similarly, consider the unique strong solution to the following system of *SDEs* in $[0,1]^n$,

$$dX_i^{(n)}(t) = 2\sqrt{X_i^{(n)}(t)(1-X_i^{(n)}(t))}dB_i^{(n)}(t) + \beta\left(a-(a+b)X_i^{(n)}(t) + \sum_{1\le j\le n, j\ne i}\frac{2X_i^{(n)}(t)(1-X_i^{(n)}(t))}{X_i^{(n)}(t)-X_j^{(n)}(t)}\right)dt, \tag{2.2}$$

where, again, the $B_i^{(n)}$, $i=1,\cdots,n$, are independent standard Brownian motions. We call this solution the β-Jacobi process. It was first introduced and studied in [54] as a generalization of the eigenvalue evolutions of matrix Jacobi processes and whose stationary distribution is given by the β-Jacobi ensemble (see Section 4 of [54]):

$$\mathcal{M}^{Jac,n}_{a,b,\beta}(dx) = C^{-1}_{n,a,b,\beta}\prod_{i=1}^n x_i^{\frac{\beta}{2}a-1}(1-x_i)^{\frac{\beta}{2}b-1}\prod_{1\le i<j\le n}|x_j-x_i|^\beta dx, \tag{2.3}$$

for some normalization constant $C_{n,a,b,\beta}$.

We now give sufficient conditions that guarantee the well-posedness of the *SDEs* above. For $\beta\ge 1$ and $d\ge 0$ and $a,b\ge 0$, (2.1) and (2.2) have a unique strong solution with no collisions and no explosions and with instant diffraction if started from a degenerate (i.e. when some of the coordinates coincide) point (see Corollary 6.5 and 6.7 respectively of [78]). In particular, the coordinates of $X^{(n)}$ stay ordered. Thus if,

$$X_1^{(n)}(0)\le\cdots\le X_n^{(n)}(0),$$

then with probability one,

$$X_1^{(n)}(t)<\cdots<X_n^{(n)}(t),\ \forall\, t>0.$$

From now on, we restrict to those parameter values.

It will be convenient to define $\theta=\frac{\beta}{2}$. We write $P^{(n)}_{d,\theta}(t)$ for the Markov semigroup associated to the solution of (2.1). Similarly, write $Q^{(n)}_{a,b,\theta}(t)$ for the Markov semigroup associated to the solution of (2.2). Furthermore, denote by $\mathcal{L}^{(n)}_{d,\theta}$ and $\mathcal{A}^{(n)}_{a,b,\theta}$ the formal infinitesimal generators for (2.1) and (2.2) respectively, given by,

$$\mathcal{L}^{(n)}_{d,\theta} = \sum_{i=1}^n 2z_i\frac{\partial}{\partial z_i^2} + 2\theta\sum_{i=1}^n\left(\frac{d}{2}+\sum_{1\le j\le n, j\ne i}\frac{2z_i}{z_i-z_j}\right)\frac{\partial}{\partial z_i}, \tag{2.4}$$

$$\mathcal{A}^{(n)}_{a,b,\theta} = \sum_{i=1}^n 2z_i(1-z_i)\frac{\partial}{\partial z_i^2} + 2\theta\sum_{i=1}^n\left(a-(a+b)z_i+\sum_{1\le j\le n, j\ne i}\frac{2z_i(1-z_i)}{z_i-z_j}\right)\frac{\partial}{\partial z_i}. \tag{2.5}$$

With I denoting either $[0,\infty)$ or $[0,1]$, define the chamber,

$$W^n(I) = \{x=(x_1,\cdots,x_n)\in I^n : x_1\le\cdots\le x_n\}.$$

Moreover, for $x \in W^{n+1}$ define the set of $y \in W^n$ that *interlace* with x by,

$$W^{n,n+1}(x) = \{y = (y_1, \cdots, y_n) \in I^n : x_1 \leq y_1 \leq x_2 \leq \cdots \leq y_n \leq x_{n+1}\}.$$

For $x \in W^{n+1}$ and $y \in W^{n,n+1}(x)$, define the *Dixon-Anderson* conditional *probability* density on $W^{n,n+1}(x)$ (originally introduced by Dixon at the beginning of the last century in [56] and independently rediscovered by Anderson in his study of the Selberg integral in [5]) by,

$$\lambda^{\theta}_{n,n+1}(x, y) = \frac{\Gamma(\theta(n+1))}{\Gamma(\theta)^{n+1}} \prod_{1 \leq i < j \leq n+1} (x_j - x_i)^{1-2\theta} \prod_{1 \leq i < j \leq n} (y_j - y_i) \prod_{i=1}^{n} \prod_{j=1}^{n+1} |y_i - x_j|^{\theta - 1}. \tag{2.6}$$

Denote by $\Lambda^{\theta}_{n,n+1}$, the integral operator with kernel $\lambda^{\theta}_{n,n+1}$ i.e.,

$$(\Lambda^{\theta}_{n,n+1} f)(x) = \int_{y \in W^{n,n+1}(x)} \lambda^{\theta}_{n,n+1}(x, y) f(y) dy.$$

Note that $\Lambda^{\theta}_{n,n+1}$ for $\theta = 1$ specializes to the Vandermonde link.

Then, our goal is to prove the following theorem, which should be considered as a generalization to the other two classical β-ensembles, the *Laguerre* and *Jacobi*, of the result of [131] for the *Gaussian* ensemble.

Theorem 2.1. *Let $\beta \geq 1$, $d \geq 2$ and $a, b \geq 1$. Then, with $\theta = \frac{\beta}{2}$, we have the following equalities of Markov kernels, $\forall t \geq 0$,*

$$P^{(n+1)}_{d-2,\theta}(t) \Lambda^{\theta}_{n,n+1} = \Lambda^{\theta}_{n,n+1} P^{(n)}_{d,\theta}(t), \tag{2.7}$$

$$Q^{(n+1)}_{a-1,b-1,\theta}(t) \Lambda^{\theta}_{n,n+1} = \Lambda^{\theta}_{n,n+1} Q^{(n)}_{a,b,\theta}(t). \tag{2.8}$$

Remark 2.2. *For $\beta = 2$, this result was already obtained in [6], see in particular subsections 3.7 and 3.8 therein respectively (also see Subsection 1.3.2 in Chapter 1).*

Remark 2.3. *The general theory of intertwining diffusions (see [125]), suggests that there should be a way to realize these intertwining relations by coupling these n and $n+1$ particle processes, so that they interlace. In the Laguerre case, (the Jacobi case is analogous) the resulting process $Z = (X, Y)$, with Y evolving according to $P^{(n)}_{d,\theta}(t)$ and X in its own filtration according to $P^{(n+1)}_{d-2,\theta}(t)$, should (conjecturally) have generator given by,*

$$\begin{aligned}\mathcal{L}^{n,n+1}_{\beta,d} = \sum_{j=1}^{n} 2y_j \partial^2_{y_j} + \beta \sum_{j=1}^{n} \left(\frac{d}{2} + \sum_{i \neq j} \frac{2y_j}{y_j - y_i} \right) \partial_{y_j} + \sum_{j=1}^{n+1} 2x_j \partial^2_{x_j} + \beta \sum_{j=1}^{n+1} \left(\frac{d-2}{2} + \sum_{i \neq j} \frac{2x_j}{x_j - x_i} \right) \partial_{x_j} \\ + (1 - \beta) \sum_{j=1}^{n+1} \sum_{i \neq j} \frac{4x_j}{x_i - x_j} \partial_{x_j} + \left(\frac{\beta}{2} - 1 \right) \sum_{j=1}^{n+1} \sum_{i=1}^{n} \frac{4x_j}{x_j - y_i} \partial_{x_j},\end{aligned}$$

with reflecting boundary conditions of the X components on the Y particles (in case they do collide). For a rigorous construction of the analogous coupled process in the case of Dyson Brownian motions *with $\beta > 2$, see Section 4 of [75].*

As just mentioned, such a coupling was constructed for Dyson Brownian motion with $\beta > 2$ in [75]; and in Chapter 1 (see also [149]) for copies of general one-dimensional diffusion processes, that in particular includes the squared Bessel (this corresponds to the Laguerre process of this chapter) and Jacobi cases for $\beta = 2$, when the interaction, between the two levels, entirely consists of local hard reflection and the transition kernels are explicit. Given such 2-level couplings, one can then iterate to construct a multilevel process in a Gelfand-Tsetlin pattern, as in [164] which initiated this program (see Proposition 1.17 also [75],[125],[6]). For a different type of coupling, for $\beta = 2$ Dyson Brownian motion, that preceded [110] and is related to the Robinson-Schensted correspondence, see [111], [113] and the related work [39].

Using Theorem 2.1 and that $\mathcal{M}^{Jac,n}_{a,b,\beta}$ is the *unique* stationary measure of (2.2) which follows from smoothness and positivity of the transition density $p^{n,\beta,a,b}_t(x,y)$, with respect to Lebesgue measure of $Q^{(n)}_{a,b,\theta}(t)$ (see Proposition 4.1 of [54]; for this to apply we further need to restrict to $a, b > \frac{1}{\beta}$) and the fact that two distinct ergodic measures must be mutually singular (see [162]), we immediately get:

Corollary 2.4. *For $\beta \geq 1$ and $a, b > 1$ and with $\theta = \frac{\beta}{2}$,*

$$\mathcal{M}^{Jac,n+1}_{a-1,b-1,\beta}\Lambda^{\theta}_{n,n+1} = \mathcal{M}^{Jac,n}_{a,b,\beta}. \tag{2.9}$$

Proof. From (2.8) we obtain that $\mathcal{M}^{Jac,n+1}_{a-1,b-1,\beta}\Lambda^{\theta}_{n,n+1}$ is the unique stationary measure of $Q^{(n)}_{a,b,\theta}(t)$ □

Before closing this introduction we remark, that in order to establish Theorem 2.1, we will follow the strategy given in [131], namely we rely on the explicit action of the generators and integral kernel on the class of Jack polynomials which, along with an exponential moment estimate, will allow us to apply the moment method. We note that, although the β-Laguerre and β-Jacobi diffusions look more complicated than β-Dyson's Brownian motion, the main computation, performed in Step 1 of the proof below, is actually simpler than the one in [131].

2.2 Preliminaries on Jack polynomials

We collect some facts on the Jack polynomials $J_\lambda(z;\theta)$ which as already mentioned will play a key role in obtaining these intertwining relations. We mainly follow [131] which in turn follows [12] (note that there is a misprint in [131]; there is a factor of $\frac{1}{2}$ missing from equation (2.7) therein c.f. equation (2.13d) in [12]). The $J_\lambda(z;\theta)$ are defined to be the (unique up to normalization) symmetric polynomial eigenfunctions in n variables of the differential operator $\mathcal{D}^{(n),\theta}$,

$$\mathcal{D}^{(n),\theta} = \sum_{i=1}^{n} z_i^2 \frac{\partial}{\partial z_i^2} + 2\theta \sum_{i=1}^{n} \sum_{1\le j\le n, j\ne i} \frac{z_i^2}{z_i - z_j} \frac{\partial}{\partial z_i}, \tag{2.10}$$

indexed by partitions $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots)$ of length l with eigenvalue $eval(\lambda, n, \theta) = 2B(\lambda') - 2\theta B(\lambda) + 2\theta(n-1)|\lambda|$ where $B(\lambda) = \sum (i-1)\lambda_i = \sum \binom{\lambda_i'}{2}$ and λ' is the conjugate partition. With 1_n denoting a row vector of n 1s, we have the normalization,

$$J_\lambda(1_n;\theta) = \theta^{-|\lambda|} \prod_{i=1}^{l} \frac{\Gamma\left(\left(n+1-i\right)\theta + \lambda_i\right)}{\Gamma\left(\left(n+1-i\right)\theta\right)}.$$

Define the following differential operators,

$$\mathcal{B}_1^{(n)} = \sum_{i=1}^{n} \frac{\partial}{\partial z_i}, \tag{2.11}$$

$$\mathcal{B}_2^{(n),\theta} = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i^2} + 2\theta \sum_{i=1}^{n} \sum_{1\le j\le n, j\ne i} \frac{z_i}{z_i - z_j} \frac{\partial}{\partial z_i}, \tag{2.12}$$

$$\mathcal{B}_3^{(n)} = \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i}. \tag{2.13}$$

Then the action of these operators on the $J_\lambda(z;\theta)$'s is given explicitly by (see [12] equations (2.13a), (2.13d) and (2.13b) respectively),

$$\mathcal{B}_1^{(n)} J_\lambda(z;\theta) = J_\lambda(1_n;\theta) \sum_{i=1}^{l} \binom{\lambda}{\lambda_{(i)}}_\theta \frac{J_{\lambda_{(i)}}(z;\theta)}{J_{\lambda_{(i)}}(1_n;\theta)}, \tag{2.14}$$

$$\mathcal{B}_2^{(n),\theta} J_\lambda(z;\theta) = J_\lambda(1_n;\theta) \sum_{i=1}^{l} \binom{\lambda}{\lambda_{(i)}}_\theta (\lambda_i - 1 + (n-i)\theta) \frac{J_{\lambda_{(i)}}(z;\theta)}{J_{\lambda_{(i)}}(1_n;\theta)}, \tag{2.15}$$

$$\mathcal{B}_3^{(n)} J_\lambda(z;\theta) = |\lambda| J_\lambda(z;\theta), \tag{2.16}$$

where $\lambda_{(i)}$ is the sequence given by $\lambda_{(i)} = (\lambda_1, \cdots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \cdots)$ (in case $i = l$ and $\lambda_i = 1$ we drop λ_l from λ) and the combinatorial coefficients $\binom{\lambda}{\rho}_\theta$ are defined by the following expansion (we set $\binom{\lambda}{\lambda_{(i)}}_\theta = 0$ in case $\lambda_{(i)}$ is no longer a non-decreasing positive sequence),

$$\frac{J_\lambda(1_n + z;\theta)}{J_\lambda(1_n;\theta)} = \sum_{m=0}^{|\lambda|} \sum_{|\rho|=m} \binom{\lambda}{\rho}_\theta \frac{J_\rho(z;\theta)}{J_\rho(1_n;\theta)},$$

but whose exact values will not be required in what follows. Finally, we need the following about the action of $\Lambda^{\theta}_{n,n+1}$ on $J_\lambda(\cdot;\theta)$ (see [116] Section 6),

$$\int_{W^{n,n+1}(x)} \lambda^{\theta}_{n,n+1}(x,y)J_\lambda(y;\theta)dy = J_\lambda(x;\theta)c(\lambda,n,\theta), \tag{2.17}$$

where,

$$c(\lambda,n,\theta) = \frac{\Gamma((n+1)\theta)}{\Gamma(\theta)} \prod_{i=1}^{n} \frac{\Gamma\left((n+1-i)\,\theta+\lambda_i\right)}{\Gamma\left((n+2-i)\,\theta+\lambda_i\right)}. \tag{2.18}$$

2.3 Proof

We split the proof in 4 steps, following the strategy laid out in [131].

Proof of Theorem 2.1. First, note that we can write the operators $\mathcal{L}^{(n)}_{d,\theta}$ and $\mathcal{A}^{(n)}_{a,b,\theta}$ as follows,

$$\mathcal{L}^{(n)}_{d,\theta} = 2\mathcal{B}^{(n),\theta}_2 + \theta d\mathcal{B}^{(n)}_1, \tag{2.19}$$

$$\mathcal{A}^{(n)}_{a,b,\theta} = 2\mathcal{B}^{(n),\theta}_2 - 2\mathcal{D}^{(n),\theta} + 2\theta a\mathcal{B}^{(n)}_1 - 2\theta(a+b)\mathcal{B}^{(n),\theta}_3. \tag{2.20}$$

Step 1 The aim of this step is to show the intertwining relation at the level of the infinitesimal generators acting on the Jack polynomials. Namely that,

$$\mathcal{L}^{(n+1)}_{d-2,\theta}\Lambda^{\theta}_{n,n+1}J_\lambda(\cdot;\theta) = \Lambda^{\theta}_{n,n+1}\mathcal{L}^{(n)}_{d,\theta}J_\lambda(\cdot;\theta), \tag{2.21}$$

$$\mathcal{A}^{(n+1)}_{a-1,b-1,\theta}\Lambda^{\theta}_{n,n+1}J_\lambda(\cdot;\theta) = \Lambda^{\theta}_{n,n+1}\mathcal{A}^{(n)}_{a,b,\theta}J_\lambda(\cdot;\theta). \tag{2.22}$$

We will show relation (2.22) for the Jacobi case and at the end of Step 1 indicate how to obtain (2.21).

(LHS)=

$$\mathcal{A}^{(n+1)}_{a-1,b-1,\theta}J_\lambda(x;\theta)c(\lambda,n,\theta) = c(\lambda,n,\theta)\left(2\mathcal{B}^{(n+1),\theta}_2 - 2\mathcal{D}^{(n+1),\theta} + 2\theta(a-1)\mathcal{B}^{(n+1)}_1 - 2\theta(a+b-2)\mathcal{B}^{(n+1),\theta}_3\right)J_\lambda(x;\theta)$$

$$= c(\lambda,n,\theta)\Big[2J_\lambda(1_{n+1};\theta)\sum_{i=1}^{l}\binom{\lambda}{\lambda_{(i)}}_\theta(\lambda_i - 1 + (n+1-i)\theta)\frac{J_{\lambda_{(i)}}(x;\theta)}{J_{\lambda_{(i)}}(1_{n+1};\theta)} - 2eval(\lambda,n+1,\theta)J_\lambda(x;\theta)$$

$$+ 2\theta(a-1)J_\lambda(1_{n+1};\theta)\sum_{i=1}^{l}\binom{\lambda}{\lambda_{(i)}}_\theta\frac{J_{\lambda_{(i)}}(x;\theta)}{J_{\lambda_{(i)}}(1_{n+1};\theta)} - 2\theta(a+b-2)|\lambda|J_\lambda(x;\theta)\Big].$$

(RHS): We start by computing $\mathcal{A}^{(n)}_{a,b,\theta} J_\lambda(y;\theta)$.

$$
\begin{aligned}
\mathcal{A}^{(n)}_{a,b,\theta} J_\lambda(y;\theta) &= \left(2\mathcal{B}_2^{(n),\theta} - 2\mathcal{D}^{(n),\theta} + 2\theta a \mathcal{B}_1^{(n)} - 2\theta(a+b)\mathcal{B}_3^{(n),\theta}\right) J_\lambda(y;\theta) \\
&= \Bigg[2J_\lambda(1_n;\theta)\sum_{i=1}^{l}\binom{\lambda}{\lambda_{(i)}}_\theta (\lambda_i - 1 + (n-i)\theta)\frac{J_{\lambda_{(i)}}(y;\theta)}{J_{\lambda_{(i)}}(1_{n+1};\theta)} - 2eval(\lambda,n,\theta)J_\lambda(y;\theta) \\
&+ 2\theta a J_\lambda(1_n;\theta)\sum_{i=1}^{l}\binom{\lambda}{\lambda_{(i)}}_\theta \frac{J_{\lambda_{(i)}}(y;\theta)}{J_{\lambda_{(i)}}(1_n;\theta)} - 2\theta(a+b)|\lambda|J_\lambda(y;\theta)\Bigg].
\end{aligned} \tag{2.23}
$$

Now, apply $\Lambda^\theta_{n,n+1}$ to obtain that,

$$
\begin{aligned}
\textbf{(RHS)} &= 2J_\lambda(1_n;\theta)\sum_{i=1}^{l}\binom{\lambda}{\lambda_{(i)}}_\theta (\lambda_i - 1 + (n-i)\theta)c(\lambda_{(i)},n,\theta)\frac{J_{\lambda_{(i)}}(x;\theta)}{J_{\lambda_{(i)}}(1_{n+1};\theta)} - 2c(\lambda,n,\theta)eval(\lambda,n,\theta)J_\lambda(x;\theta) \\
&+ 2\theta a J_\lambda(1_n;\theta)\sum_{i=1}^{l}\binom{\lambda}{\lambda_{(i)}}_\theta c(\lambda_{(i)},n,\theta)\frac{J_{\lambda_{(i)}}(x;\theta)}{J_{\lambda_{(i)}}(1_n;\theta)} - 2\theta(a+b)|\lambda|c(\lambda,n,\theta)J_\lambda(x;\theta).
\end{aligned}
$$

Now, in order to check **(LHS)=(RHS)** we check that the coefficients of J_λ and $J_{\lambda_{(i)}}$ $\forall i$ coincide on both sides.

- First, the coefficients of $J_\lambda(x;\theta)$:
(LHS): $-2c(\lambda,n,\theta)eval(\lambda,n+1,\theta) - c(\lambda,n,\theta)|\lambda|2\theta(a+b-2)$.
(RHS): $-2c(\lambda,n,\theta)eval(\lambda,n,\theta) - c(\lambda,n,\theta)|\lambda|2\theta(a+b)$.
These are equal iff:

$$
\frac{-2eval(\lambda,n,\theta) + 2eval(\lambda,n+1,\theta)}{4\theta|\lambda|} = 1,
$$

which is easily checked from the explicit expression of $eval(n,\lambda,\theta)$.

- Now, for the coefficients of $J_{\lambda_{(i)}}(x;\theta)$:
(LHS):

$$
2J_\lambda(1_{n+1};\theta)\binom{\lambda}{\lambda_{(i)}}_\theta (\lambda_i - 1 + (n+1-i)\theta)\frac{c(\lambda,n,\theta)}{J_{\lambda_{(i)}}(1_{n+1};\theta)} + 2\theta(a-1)J_\lambda(1_{n+1};\theta)\binom{\lambda}{\lambda_{(i)}}_\theta \frac{c(\lambda,n,\theta)}{J_{\lambda_{(i)}}(1_{n+1};\theta)}.
$$

(RHS):

$$
2J_\lambda(1_n;\theta)\binom{\lambda}{\lambda_{(i)}}_\theta (\lambda_i - 1 + (n-i)\theta)\frac{c(\lambda_{(i)},n,\theta)}{J_{\lambda_{(i)}}(1_n;\theta)} + 2\theta a J_\lambda(1_n;\theta)\binom{\lambda}{\lambda_{(i)}}_\theta \frac{c(\lambda_{(i)},n,\theta)}{J_{\lambda_{(i)}}(1_n;\theta)}.
$$

These are equal iff:

$$a-1=\frac{J_\lambda(1_n;\theta)c(\lambda_{(i)},n,\theta)J_{\lambda_{(i)}}(1_{n+1};\theta)}{J_{\lambda_{(i)}}(1_n;\theta)c(\lambda,n,\theta)J_\lambda(1_{n+1};\theta)}a+\frac{1}{\theta}\frac{J_\lambda(1_n;\theta)c(\lambda_{(i)},n,\theta)J_{\lambda_{(i)}}(1_{n+1};\theta)}{J_{\lambda_{(i)}}(1_n;\theta)c(\lambda,n,\theta)J_\lambda(1_{n+1};\theta)}(\lambda_i-1+(n-i)\theta)$$
$$-\frac{1}{\theta}(\lambda_i-1+(n+1-i)\theta).$$

We first claim that,

$$\frac{J_\lambda(1_n;\theta)c(\lambda_{(i)},n,\theta)J_{\lambda_{(i)}}(1_{n+1};\theta)}{J_{\lambda_{(i)}}(1_n;\theta)c(\lambda,n,\theta)J_\lambda(1_{n+1};\theta)}=1.$$

This immediately follows from,

$$\begin{aligned}
\frac{J_\lambda(1_n;\theta)}{J_{\lambda_{(i)}}(1_n;\theta)}&=\theta^{-1}\frac{\Gamma\left((n+1-i)\,\theta+\lambda_i\right)}{\Gamma\left((n+1-i)\,\theta+\lambda_i-1\right)},\\
\frac{J_{\lambda_{(i)}}(1_{n+1};\theta)}{J_\lambda(1_{n+1};\theta)}&=\theta\frac{\Gamma\left((n+2-i)\,\theta+\lambda_i-1\right)}{\Gamma\left((n+2-i)\,\theta+\lambda_i\right)},\\
\frac{c(\lambda_{(i)},n,\theta)}{c(\lambda,n,\theta)}&=\frac{\Gamma\left((n+1-i)\,\theta+\lambda_i-1\right)\Gamma\left((n+2-i)\,\theta+\lambda_i\right)}{\Gamma\left((n+1-i)\,\theta+\lambda_i\right)\Gamma\left((n+2-i)\,\theta+\lambda_i-1\right)}.
\end{aligned}$$

Hence, we need to check that the following is true,

$$a-1=a+\frac{1}{\theta}(\lambda_i-1+(n-i)\theta)-\frac{1}{\theta}(\lambda_i-1+(n-i+1)\theta),$$

which is obvious.

Now, in order to obtain (2.21) we only need to consider coefficients in $J_{\lambda_{(i)}}$'s (since the operators $\mathcal{D}^{(n),\theta}$ and $\mathcal{B}_3^{(n)}$ that produce J_λ's are missing) and replace a by $\frac{d}{2}$.

To prove the analogous result for β-Dyson Brownian motions, one needs to observe, as done in [131], that the generator of n particle β-Dyson Brownian motion $L_\theta^{(n)}$ can be written as a commutator, namely $L_\theta^{(n)}=[\mathcal{B}_1^{(n)},\mathcal{B}_2^{(n),\theta}]=\mathcal{B}_1^{(n)}\mathcal{B}_2^{(n),\theta}-\mathcal{B}_2^{(n),\theta}\mathcal{B}_1^{(n)}$.

Step 2 We obtain an exponential moment estimate, namely regarding $\mathbb{E}_x\left[e^{\epsilon\|X^{(n)}(t)\|}\right]$. This is obviously finite by compactness of $[0,1]^n$ in the Jacobi case. In the Laguerre case, we proceed as follows. Writing $X^{(n)}$ for the solution to (2.1), letting $\|\cdot\|$ denote the l_1 norm and recalling that all entries of $X^{(n)}$ are non-negative we obtain,

$$d\|X^{(n)}(t)\|=\sum_{i=1}^n 2\sqrt{dX_i^{(n)}(t)}dB_i^{(n)}(t)+\beta\left(\frac{d}{2}n+\sum_{i=1}^n\sum_{1\le j\le n, j\ne i}\frac{2X_i^{(n)}(t)}{X_i^{(n)}(t)-X_j^{(n)}(t)}\right)dt.$$

Note that,

$$\sum_{i=1}^n\sum_{1\le j\le n, j\ne i}\frac{2X_i^{(n)}(t)}{X_i^{(n)}(t)-X_j^{(n)}(t)}=2\binom{n}{2},$$

and that by Levy's characterization the local martingale $(M(t), t \geq 0)$ defined by,

$$dM(t) = \frac{1}{\sqrt{\|X^{(n)}(t)\|}} \sum_{i=1}^{n} \sqrt{X_i^{(n)}(t)} dB_i^{(n)}(t),$$

is equal to a standard Brownian motion $(W(t), t \geq 0)$ and so we obtain,

$$d\|X^{(n)}(t)\| = 2\sqrt{\|X^{(n)}(t)\|} dW(t) + \beta \left(\frac{d}{2} n + 2\binom{n}{2} \right) dt.$$

Thus, $\|X^{(n)}(t)\|$ is a squared Bessel process of dimension $dim_{\beta,n,d} = \beta\left(\frac{d}{2}n + 2\binom{n}{2}\right)$. Hence, from standard estimates (see [134] Chapter IX.1 or Proposition 2.1 of [59]; in case that $dim_{\beta,n,d}$ is an integer the result is an immediate consequence of Fernique's theorem ([70]) since $\|X^{(n)}(t)\|$ is the square of a Gaussian process) it follows that, for $\epsilon > 0$ small enough, $\mathbb{E}_x\left[e^{\epsilon\|X^{(n)}(t)\|}\right] < \infty$.

Step 3 We now lift the intertwining relation to the semigroups acting on the Jack polynomials, namely,

$$\begin{aligned} P_{d-2,\theta}^{(n+1)}(t)\Lambda_{n,n+1}^{\theta} J_\lambda(\cdot;\theta) &= \Lambda_{n,n+1}^{\theta} P_{d,\theta}^{(n)}(t) J_\lambda(\cdot;\theta), \\ Q_{a-1,b-1,\theta}^{(n+1)}(t)\Lambda_{n,n+1}^{\theta} J_\lambda(\cdot;\theta) &= \Lambda_{n,n+1}^{\theta} Q_{a,b,\theta}^{(n)}(t) J_\lambda(\cdot;\theta). \end{aligned}$$

The proof follows almost word for word the elegant argument given in [131]. We reproduce it here, elaborating a bit on some parts, for the convenience of the reader, moreover only considering the Laguerre case for concreteness. We begin by applying Ito's formula to $J_\lambda(X^{(n)}(t);\theta)$ and taking expectations (note that the stochastic integral term is a true martingale since its expected quadratic variation is finite which follows by the exponential estimate of Step 2) we obtain,

$$P_{d,\theta}^{(n)}(t) J_\lambda(\cdot;\theta) = J_\lambda(\cdot;\theta) + \int_0^t P_{d,\theta}^{(n)}(s) \mathcal{L}_{d,\theta}^{(n)} J_\lambda(\cdot;\theta) ds. \tag{2.24}$$

Now, note that by (2.23), $\mathcal{L}_{d,\theta}^{(n)} J_\lambda(\cdot;\theta)$ is given by a linear combination of Jack polynomials $J_\kappa(\cdot;\theta)$ for some partitions κ with $\kappa_i \leq \lambda_i \ \forall i \leq l$ and we will write $\kappa \leq \lambda$ if this holds. We will denote the action of $\mathcal{L}_{d,\theta}^{(n)}$ on this *finite* dimensional vector space, spanned by the Jack polynomials indexed by partitions κ with $\kappa \leq \lambda$, by the matrix M_2.

Moreover, each $J_\kappa(\cdot;\theta)$ for $\kappa \leq \lambda$ obeys (2.24) and thus we obtain the following system of integral equations, with $f_\kappa(t) = P_{d,\theta}^{(n)}(t) J_\kappa(\cdot;\theta)$,

$$f_\kappa(t) = f_\kappa(0) + \sum_{\nu \leq \lambda} M_2(\kappa,\nu) \int_0^t f_\nu(s) ds,$$

whose unique solution is given by the matrix exponential,

$$f_\kappa(t) = \sum_{\nu \le \lambda} e^{tM_2}(\kappa, \nu) f_\nu(0). \tag{2.25}$$

Now, observe that by (2.17) the Markov kernel $\Lambda^\theta_{n,n+1}$ also acts on the aforementioned finite dimensional vector space of Jack polynomials as a matrix, which we denote by M_1. We will also denote by a matrix M_3 the action of $\mathcal{L}^{(n+1)}_{d-2,\theta}$ and note that the intertwining relation (2.21) can be written in terms of matrices as follows: $M_3 M_1 = M_1 M_2$. Thus, making use of the following elementary fact about finite dimensional square matrices,

$$M_3 M_1 = M_1 M_2 \implies e^{tM_3} M_1 = M_1 e^{tM_2} \text{ for } t \ge 0,$$

and display (2.25), along with its analogue with M_2 replaced by M_3, we get that,

$$P^{(n+1)}_{d-2,\theta}(t) \Lambda^\theta_{n,n+1} J_\lambda(\cdot;\theta) = \Lambda^\theta_{n,n+1} P^{(n)}_{d,\theta}(t) J_\lambda(\cdot;\theta).$$

Step 4 We again follow [131]. Recall, (see [131] and the references therein) that we can write any *symmetric* polynomial p in n variables as a finite linear combination of Jack polynomials in n variables. Hence, for any such p,

$$P^{(n+1)}_{d-2,\theta}(t) \Lambda^\theta_{n,n+1} p(\cdot) = \Lambda^\theta_{n,n+1} P^{(n)}_{d,\theta}(t) p(\cdot), \tag{2.26}$$

$$Q^{(n+1)}_{a-1,b-1,\theta}(t) \Lambda^\theta_{n,n+1} p(\cdot) = \Lambda^\theta_{n,n+1} Q^{(n)}_{a,b,\theta}(t) p(\cdot). \tag{2.27}$$

Now, any probability measure μ on $W^n(I)$ will give rise to a symmetrized probability measure μ^{symm} on I^n as follows,

$$\mu^{symm}(dz_1 . \cdots, dz_n) = \frac{1}{n!} \mu(dz_{(1)} . \cdots, dz_{(n)}),$$

where $z_{(1)} \le z_{(2)} \le \cdots \le z_{(n)}$ are the order statistics of $(z_1, z_2, \cdots, z_n)$. Moreover, for every (not necessarily symmetric) polynomial q in n variables, with S_n denoting the symmetric group on n symbols, we have,

$$\int_{I^n} q(z) d\mu^{symm}(z) = \int_{I^n} \frac{1}{n!} \sum_{\sigma \in S_n} q(z_{\sigma(1)}, \cdots, z_{\sigma(n)}) d\mu^{symm}(z) = \int_{W^n(I)} \frac{1}{n!} \sum_{\sigma \in S_n} q(z_{\sigma(1)}, \cdots, z_{\sigma(n)}) d\mu(z).$$

Note that now $p(z) = \frac{1}{n!} \sum_{\sigma \in S_n} q(z_{\sigma(1)}, \cdots, z_{\sigma(n)})$ is a symmetric polynomial (in n variables). Thus, from (2.26) and (2.27) all moments of the symmetrized versions of both sides of (2.7) and (2.8) coincide. Hence, by Theorem 1.3 of [84] (and the discussion following it) along with the fact that $(\Lambda^\theta_{n,n+1} f)(z) \le e^{\epsilon \|z\|_1}$ where $f(y) = e^{\epsilon \|y\|_1}$ (since all coordinates are positive) and our exponential moment estimate from Step 2 we obtain that the symmetrized versions of both sides of (2.7) and (2.8) coincide; where we view for each $x \in W^{n+1}$ and $t \ge 0$ $P^{(n+1)}_{d-2,\theta}(t) \Lambda^\theta_{n,n+1}$ and $\Lambda^\theta_{n,n+1} P^{(n)}_{d,\theta}(t)$ as probability measures on W^n. In fact, by the discussion after Theorem 1.3 of [84], since we work in $[0,\infty)^n$ and not the full space $\mathbb{R}^n$, we need not require that the symmetrized versions of these measures have exponential moments but that they only need to integrate $e^{\epsilon\sqrt{\|z\|}}$. The theorem is now proven. $\square$

Chapter 3

Feller processes on the graph of spectra and the Hua-Pickrell measures

3.1 Introduction

The main result of this chapter is the construction of a Feller-Markov process on the infinite dimensional boundary Ω of the *"graph of spectra"*, the continuum analogue of the classical Gelfand-Tsetlin graph, leaving the *Hua-Pickrell measures* on Ω invariant, by the so called *method of intertwiners*.

This approach, of constructing such Feller processes, was first introduced by Borodin and Olshanski in [28] in order to obtain stochastic dynamics on the boundary of the Gelfand-Tsetlin graph, which describes the branching of irreducible representations of the chain of unitary groups $\mathbb{U}(1) \subset \mathbb{U}(2) \subset \cdots$, that leave the zw measures invariant; these distinguished measures first arose in the problem of the harmonic analysis on the infinite dimensional unitary group $\mathbb{U}(\infty)$, see in particular [120] for more details.

The formalism of the intertwiners was then subsequently successfully applied in the case of the infinite symmetric group $S(\infty)$ in [29] where in fact a more complete study of the properties of the resulting infinite dimensional process is possible (in particular its space-time correlation kernels can be computed explicitly) and also very recently by Cuenca in [50] for the BC-type branching graph, which is related to the infinite symplectic $Sp(\infty)$ and orthogonal $O(\infty)$ groups (see chapter 5 for a detailed study of these problems).

However, until now all these applications have been in the discrete setting and this contribution is the first one that deals directly with the continuum. Moreover, it should be noted that in the random matrix setting this is the first time an infinite dimensional Markov process is constructed starting from an arbitrary initial configuration and having the Feller property. Even in the simpler model of Dyson Brownian motion, in the works of Osada (see for example [124] and references therein) only equilibrium dynamics are considered and also in the tour de force work of Tsai [153] the initial configuration needs to satisfy a certain balanced assumption. As will become clear, the reason we can achieve this construction is because we take advantage of all integrable structures underlying this problem. Finally, as the Gelfand-Tsetlin graph degenerates to the graph of spectra under a limiting transition, we expect the dynamics constructed in this chapter to be naturally related through a scaling limit (after possibly scaling the parameters as well) with the dynamics considered in [28], although the exact connection remains mysterious for now, see section 3.6.

We now proceed to give a more detailed, although still informal, exposition of our results. All notions introduced below will be made precise in the relevant sections later on.

We begin in section 3.2 by recalling several facts about unitarily invariant measures on the space of infinite Hermitian matrices H; these are precisely the measures invariant under the action by conjugation of $\mathbb{U}(\infty)$. As in all the settings mentioned above, these measures have a representation theoretic meaning as well, the ergodic invariant measures are in one to one correspondence with (equivalence classes of) spherical representations (T,ξ) of the infinite dimensional Cartan motion group $G(\infty)=\lim_{N\to\infty} G(N)$ where $G(N)=\mathbb{U}(N)\ltimes H(N)$, the reader is referred to [127] and [123] for more details. The fundamental and indeed very remarkable result in the area, first appearing in Vershik's note [158] where he introduced the so called *ergodic method*, later also proved by Pickrell [127] and a more detailed exposition of the original proof of Vershik appearing in [123], is the fact that the extremal or ergodic $\mathbb{U}(\infty)$ invariant measures can be characterized explicitly and are parametrized by an infinite dimensional space Ω defined in (3.1).

We then define the "graph of spectra", which is not really a graph in the rigorous sense (that explains our use of quotation marks), but rather a projective chain. This is given by the sequence $\{W^N\}_{N\ge1}$ of Weyl chambers in $\mathbb{R}^N$ namely $(x_1,\cdots,x_N)\in W^N$ if $x_1\le\cdots\le x_N$ and Markov kernels or links $\Lambda_N^{N+1}:W^{N+1}\to W^N$ given by ratios of Vandermonde determinants $\Delta_N(x)=\prod_{1\le i<j\le N}(x_j-x_i)$ as follows (some slight care is needed when some of x coordinates coincide see Section 2.2),

$$\Lambda_N^{N+1}(x,dy)=\frac{N!\Delta_N(y)}{\Delta_{N+1}(x)}1(y\prec x)dy,$$

where for $y\in W^N, x\in W^{N+1}$ $y\prec x$ denotes interlacing: $x_1\le y_1\le x_2\le\cdots\le x_{N+1}$. It turns out that the Feller boundary in the sense of Borodin and Olshanski of this chain can be identified with the space Ω. More precisely (but note that this is not the exact definition of a Feller boundary, some extra conditions are needed), the extreme set of the convex set consisting of sequences of coherent *probability* measures $\{\mu_N\}_{N\ge1}$ on $\{W^N\}_{N\ge1}$ namely so that,

$$\mu_{N+1}\Lambda_N^{N+1}=\mu_N\ ,\forall N\ge1,$$

can be parametrized by Ω; moreover the Markov kernels $\Lambda_N^\infty:\Omega\to W^N\ \forall N\ge1$ (under certain regularity assumptions) are given explicitly in terms of a single *totally positive* *function*. We then close this section, following [28] with a brief introduction to the main results of the method of intertwiners that we will use later on.

In section 3.3, we introduce the Hua-Pickrell measures $\mu_{HP}^{s,N}$ on W^N, where s is a complex parameter, that will be our main focus in this work. These measures were first studied by Hua Luogeng in the 50's in his book [81] on harmonic analysis in several complex variables and were later in the 80's rediscovered independently by Pickrell [128] in the context of Grassmann manifolds. Then, around the turn of the millennium, Neretin studied a generalization as part of a larger program in [108] and Borodin and Olshanski investigated in particular their determinantal properties [25]. Very recently, in the last few years, there has been a lot of activity around these measures, also in the infinite case (when they can no longer be normalized to be probability measures) and several open problems have been settled by Bufetov and Qiu (see for example [43] and [130] and the references therein). We will collect several of their properties and key facts, the most fundamental being that they are consistent with the links Λ_N^{N+1} above,

$$\mu_{HP}^{s,N+1}\Lambda_N^{N+1} = \mu_{HP}^{s,N}, \forall N \geq 1,$$

so that in particular we obtain, a non-extremal or equivalently not a delta function, measure μ_{HP}^{s} on Ω. We mention in passing that, we will also give an independent proof of the consistency relation above, that avoids any difficult explicit computations of integrals, using the dynamical approach advocated in this chapter.

In section 3.4 we introduce our stochastic dynamics. Akin to the classical case of Dyson's Brownian motion for $\beta = 2$ these are given equivalently as a Doob's h-transform of one dimensional diffusions (with transition densities in $\mathbb{R}$ denoted by $p_t^{(N),s}$) killed when they intersect i.e. with transition density in $\mathring{W}^N$ given by,

$$e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)}\det\left(p_t^{(N),s}(x_i, y_j)\right)_{i,j=1}^N dy,$$

or as the unique strong solution to the system of Stochastic Differential Equations (*SDEs*) with long range repulsion where the $\{W_i\}_{i=1}^N$ are independent standard Brownian motions,

$$dX_i(t) = \sqrt{2(X_i^2(t)+1)}dW_i(t) + \left[(2-2N-2\Re(s))\,X_i(t) + 2\Im(s) + \sum_{j\neq i}\frac{2(X_i^2(t)+1)}{X_i(t)-X_j(t)}\right]dt.$$

We prove well-posedness and the Feller property for these processes and most importantly, that for $\Re(s) > -\frac{1}{2}$ the measures $\mu_{HP}^{s,N}$ are their unique invariant probability measures. Namely, if we denote by $P_{HP}^{s,N}(t)$ the Feller semigroups associated to the processes above we have for $\Re(s) > -\frac{1}{2}$,

$$\mu_{HP}^{s,N}P_{HP}^{s,N}(t) = \mu_{HP}^{s,N} \;\; t \geq 0, \forall N \geq 1.$$

We then arrive at section 3.5 where, after recalling some necessary results from Chapter 1 where intertwining relations between determinantal semigroups are studied, we prove our main result, the following consistency relation between the semigroups,

$$P_{HP}^{s,N+1}(t)\Lambda_N^{N+1} = \Lambda_N^{N+1} P_{HP}^{s,N}(t) \ , \ t \geq 0, \forall N \geq 1.$$

We thus, via the formalism of the method of intertwiners, obtain a Feller-Markov process with semigroup $P_{HP}^{s,\infty}(t)$ on Ω consistent with the stochastic processes on level N,

$$P_{HP}^{s,\infty}(t)\Lambda_N^{\infty} = \Lambda_N^{\infty} P_{HP}^{s,N}(t) \ , \ t \geq 0, \forall N \geq 1,$$

that has μ_{HP}^s for $\Re(s) > -\frac{1}{2}$ as its unique invariant probability measure. Since the description of these processes might seem a bit abstract and out of reach, we then discuss a hands on approximation procedure for boundary Feller processes from their finite N analogues. Furthermore, as is by now relatively well known there are other (except the Hua-Pickrell introduced here) multidimensional diffusions consistent with the links Λ_N^{N+1}. The two most classical and simplest examples being Dyson's Brownian motion and its stationary Ornstein-Uhlenbeck counterpart (see for example [164] and for general β [131], also the first chapter of this thesis). By the intertwiners formalism, one again obtains a Feller process for each on Ω. It turns out however that, these processes are simple deterministic dynamical systems and we showcase this by the rather down to earth approximation procedure mentioned above, see subsection 3.5.2 for more details.

Moving on to section 3.6, we make the connection to interacting particle systems in $(2+1)$-dimensions. The motivation behind this section is to provide a relation with the discrete dynamics considered by Borodin and Olshanski on the path space of the Gelfand-Tsetlin graph. More precisely, making use of the general results of chapter 1, we construct consistent dynamics on the *path space* of the graph of spectra leaving the multilevel Hua-Pickrell measures invariant. This path space is given equivalently by infinite interlacing arrays. More specifically, a path of length N is given by a continuous Gelfand-Tsetlin pattern of depth N, denoted by $\mathbb{GT}_c(N)$. The diffusion processes $\mathbb{X}^{(N)}$ we construct in $\mathbb{GT}_c(N)$ (note that there must be some interaction between the components in order for the interlacing to remain) are such that if they are started according to a *Gibbs* or *Central* measure (see display (3.14) for a precise definition) then the projection $\pi_n \mathbb{X}^{(N)} = \left(X_1^{(n)}, \cdots, X_n^{(n)}\right)$ on the n^{th} level evolves according to $P_{HP}^{s,n}(t)$.

Then, in section 3.7 we study how our results transfer to the circle $\mathbb{T}$ under an application of the *Cayley* transform, which in more generality maps Hermitian matrices to unitary matrices. For the particular case $s = 0$, we obtain an interlacing process that leaves the multilevel Circular Unitary Ensemble (CUE) invariant.

Finally, in the short concluding section 3.8 we introduce a matrix valued (more precisely Hermitian valued) process whose eigenvalue evolution is that of the system of *SDEs* considered above. This matrix evolution will be studied in detail in the next chapter.

3.2 Ergodic measures and the boundary of the graph of spectra

3.2.1 Ergodic unitarily invariant measures

We begin by recalling some useful facts about unitarily invariant measures on the space of infinite Hermitian matrices. We mainly follow [25] and [123],the connection to the graph of spectra will be clarified in the sequel. So, let $\mathbb{U}(N)$ be the N dimensional unitary group. Let $H(N)$ denote the space of $N \times N$ Hermitian matrices. Define the Cayley transform that maps $X \in H(N)$ to $U \in \mathbb{U}(N)$ by,

$$X \mapsto U = \frac{i - X}{i + X}.$$

We denote this bijective map by $\mathfrak{C}$ and by π_N^{N+1} the "cutting corner" projection from $H(N+1)$ to $H(N)$: $\pi_N^{N+1}\left[\left(h_{ij}\right)_{i,j=1}^{N+1}\right] = \left(h_{ij}\right)_{i,j=1}^{N}$. Finally we will write $\mathsf{eval}_N : H(N) \to W^N$ for the map on Hermitian matrices $H(N)$ defined by $\mathsf{eval}_N(H) = (x_1 \le \cdots \le x_N)$ where the $(x_1 \le \cdots \le x_N)$ are the ordered eigenvalues of the matrix H.

Moving on, we let H denote the projective limit $\varprojlim H(n)$, the space of all infinite Hermitian matrices which can be naturally identified as a topological vector space with the infinite product $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ by:

$$H \ni X \mapsto \{X_{ii}\} \sqcup \{\Re X_{ij}, \Im X_{ij}\}.$$

Moreover, let $H(\infty)$ denote the inductive limit, $\lim_{N\to\infty} H(N)$, the space of $\infty \times \infty$ Hermitian matrices with finitely many non-zero entries and similarly we consider $\mathbb{U}(\infty) = \lim_{N\to\infty} \mathbb{U}(N)$ the inductive limit unitary group. With these definitions in place, there exists a pairing,

$$H(\infty) \times H \to \mathbb{R}\,, \;\; (A, X) \mapsto Tr(AX).$$

Now, for a Borel probability measure M on H define its Fourier transform as the function on $H(\infty)$ denoted by,

$$\hat{M}(A) = F_M(A) = \int_H e^{iTr(AX)} M(dX) \;\text{ for }\; A \in H(\infty).$$

The group $\mathbb{U}(\infty)$ acts on both $H(\infty)$ and H by conjugation and the pairing of the two spaces is $\mathbb{U}(\infty)$ invariant. Observe that a matrix in $H(\infty)$ can be brought by conjugation to a diagonal matrix $\mathrm{diag}(r_1, r_2, \cdots)$ with finitely many non-zero entries. Thus, the Fourier transform of $\mathbb{U}(\infty)$ invariant probability measures on H, that we denote by $\mathcal{M}_p^{\mathbb{U}(\infty)-inv}(H)$, is uniquely determined by its values on the diagonal matrices from $H(\infty)$. It is a remarkable fact that, extremal or ergodic $\mathbb{U}(\infty)$ (these notions are of course equivalent see for example

Proposition 1.3 of [123]) invariant probability measures, $\mathcal{M}_p^{\mathbb{U}(\infty)-erg}(H)$, can be explicitly characterized. Define the space Ω by,

$$\begin{aligned}
&\Omega = \Big\{\omega = (\alpha^+,\alpha^-,\gamma_1,\delta) \in \mathbb{R}^{2\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R} | \\
&\alpha^+ = (\alpha_1^+ \geq \alpha_2^+ \geq \cdots \geq 0) \;\; \alpha^- = (\alpha_1^- \geq \alpha_2^- \geq \cdots \geq 0) \\
&\gamma_1 \in \mathbb{R} \;\; \sum (\alpha_i^+)^2 + \sum (\alpha_i^-)^2 \leq \delta \Big\}
\end{aligned} \tag{3.1}$$

and moreover let $\gamma_2 = \delta - \sum(\alpha_i^+)^2 - \sum(\alpha_i^-)^2$. We note that Ω is a locally compact metrizable topological space with a countable base. Finally, write F_ω for,

$$F_\omega(x) = e^{i\gamma_1 x - \frac{\gamma_2}{2}x^2} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k^+ x}}{1 - i\alpha_k^+ x} \prod_{k=1}^{\infty} \frac{e^{i\alpha_k^- x}}{1 + i\alpha_k^- x}.$$

Observe that we have the estimate:

$$\frac{e^{iax}}{1 - iax} = 1 - \frac{3}{2}a^2x^2 + O\left(a^3\right) \;\text{ as } a \to 0.$$

Thus, since $\sum(\alpha_i^+)^2 + \sum(\alpha_i^-)^2 < \infty$ the function $F_\omega(x)$ converges for all $x \in \mathbb{R}$ for any $\omega \in \Omega$; with the result being a continuous function. Moreover, observe that for any fixed $x \in \mathbb{R}$, $F_\omega(x)$ as a function of $\omega \in \Omega$ is continuous.

The following fundamental Theorem was first stated and a proof was outlined by Vershik in [158]. It was later also proven by Pickrell [127] by exploiting the connection to total positivity. A more detailed exposition of the original method of [158] was subsequently given by Olshanski and Vershik in [123], (see also Defosseux [53]).

Theorem 3.1. *There exists a parametrization of ergodic/extremal $\mathbb{U}(\infty)$-invariant probability measures on the space H, $\mathcal{M}_p^{\mathbb{U}(\infty)-erg}(H)$, by the points of the space Ω. Given ω the characteristic function of the ergodic measure M_ω is given by,*

$$\int_{X \in H} e^{iTr(\mathrm{diag}(r_1,\cdots,r_n,0,0,\cdots)X)} M_\omega(dX) = \prod_{j=1}^{n} F_\omega(r_j).$$

Remark 3.2. *We observe that the characteristic function F_ω of an ergodic measure M_ω is given as a product of characteristic functions of simpler measures, with only one non-zero parameter, that we call elementary. Equivalently any ergodic measure is given as a convolution of elementary ergodic ones. More precisely writing this in terms of a sum of independent random Hermitian matrices:*

$$\gamma_1 \mathrm{Id} + G^{\gamma_2} + \sum_{k \geq 1} \alpha_k^+ \left[-\mathrm{Id} + \zeta^*(k)\zeta(k)\right] + \sum_{k \geq 1} \left(-\alpha_k^-\right)\left[-\mathrm{Id} + \xi^*(k)\xi(k)\right].$$

Here, G^{γ_2} is an infinite GUE matrix, namely the entries $G_{ii}^{\gamma_2}$ and $\Re G_{ij}^{\gamma_2}$, $\Im G_{ij}^{\gamma_2}$ are independent normal random variables of mean 0 and variance γ_2 subject to the Hermitian constraint. Moreover, *the $\zeta(k)$ and $\xi(k)$ are independent infinite row vectors whose entries are i.i.d. complex normal random variables. For more details see Remarks 2.10-2.13 of [123] and also Defosseux [53] Theorem 2.7.*

The following notion will be very useful in what follows. We call a *real smooth non-negative* function $\phi(x)$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} \phi(x)dx = 1$ *extended totally positive* if,

$$\det\left(\phi^{(i-1)}(x_{n+1-j})\right)_{i,j=1}^{n} \geq 0\ ,\ n = 1, 2, \cdots \text{ and } x_1 < \cdots < x_n.$$

Then, by Theorem 7.7 of [123] (see also Proposition 7.6 part (ii) therein) for $\omega \in \Omega$ with $\gamma_2(\omega) > 0$ a function ϕ such that its Fourier transform is given by,

$$\hat{\phi}(\xi) = \hat{\phi}_{\omega}(\xi) = F_{\omega}(\xi),$$

is extended totally positive. In fact, for $\gamma_2(\omega) > 0$ the inequalities above are strict (see Proposition 7.6 part (ii) of [123]) and so (by Theorem 2.1 page 50 of [91]) ϕ_{ω} is *totally positive* namely for $n \geq 1$ and $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$,

$$\det\left(\phi_{\omega}(x_i - y_j)\right)_{i,j=1}^{n} \geq 0.$$

3.2.2 The graph of Spectra and its Boundary

We start by setting up some notation. Write $x = (x_1, \cdots, x_N) \in W^N$ if $x_1 \leq \cdots \leq x_N$ and furthermore write $W^{N,N+1}(x)$ for the set of $y \in W^N$ that interlace with $x \in W^{N+1}$ i.e. $x_1 \leq y_1 \leq x_2 \leq \cdots \leq y_N \leq x_{N+1}$ (we will also denote this suppressing any dependence on N by $y \prec x$). We define the Markov kernel for $x \in \mathring{W}^{N+1}$, the interior of W^{N+1},

$$\Lambda_N^{N+1}(x, dy) = \frac{N!\Delta_N(y)}{\Delta_{N+1}(x)} 1(y \in W^{N,N+1}(x))dy,$$

where $\Delta_N(y) = \prod_{1 \leq i < j \leq N}(y_j - y_i)$. In fact, the Markov kernel above has an interpretation as a conditional distribution for matrix eigenvalues, the first published proof of this fact was given by Baryshnikov (see Proposition 4.2 in [14]) in the random matrix literature (see also Proposition 3.1 in [121] and the historical comments therein). Namely, it is the law of:

$$\mathsf{eval}_N\left(\pi_N^{N+1}\left[U^* \mathrm{diag}\left(x_1, \cdots, x_{N+1}\right) U\right]\right) \tag{3.2}$$

where U is a Haar distributed unitary matrix from $\mathbb{U}(N+1)$. Observe that the expression (3.2) makes sense for arbitrary $x \in W^{N+1}$. Thus, for any $x \in W^{N+1}$ we take as the definition of $\Lambda_N^{N+1}(x, \cdot)$ the law of (3.2).

We will see in the proof of the Lemma below that this definition coincides with the weak limit of $\Lambda_N^{N+1}\left(x^{(n)}, \cdot\right)$ for $\{x^{(n)}\}_n \in \mathring{W}^{N+1}$ converging to x. Denote by $C_0\left(W^N\right)$ the space of continuous functions on W^N vanishing at infinity.

Lemma 3.3. *Then, Λ_N^{N+1} is a Feller kernel i.e.,*

$$\Lambda_N^{N+1} f \in C_0\left(W^{N+1}\right), \forall f \in C_0\left(W^N\right).$$

Proof. We have:

$$\left[\Lambda_N^{N+1} f\right](x_1, \cdots, x_{N+1}) = \mathbb{E}_{\mathbb{U}(N+1)}\left[f\left[\mathsf{eval}_N\left(\pi_N^{N+1}\left[U^*\mathrm{diag}\left(x_1, \cdots, x_{N+1}\right)U\right]\right)\right]\right].$$

Thus, if we take any sequence $x^{(n)} \in W^{N+1}$ converging to some $x \in W^{N+1}$ by the dominated convergence theorem and continuity of all functions involved in the representation above we obtain:

$$\left[\Lambda_N^{N+1} f\right](x_1^{(n)}, \cdots, x_{N+1}^{(n)}) \to \left[\Lambda_N^{N+1} f\right](x_1, \cdots, x_{N+1}).$$

In particular, we have the weak convergence of probability measures:

$$\Lambda_N^{N+1}(x^{(n)}, \cdot) \rightharpoonup \Lambda_N^{N+1}(x, \cdot).$$

Now, we show that as $x^{(n)} \to \infty$ we have $\left[\Lambda_N^{N+1} f\right](x_1^{(n)}, \cdots, x_{N+1}^{(n)}) \to 0$. Without loss of generality assume $x_{N+1}^{(n)} \to \infty$. If $x_N^{(n)} \to \infty$ as well, necessarily by interlacing of eigenvalues we have:

$$\mathsf{eval}_N\left(\pi_N^{N+1}\left[U^*\mathrm{diag}\left(x_1, \cdots, x_{N+1}\right)U\right]\right) \to \infty.$$

Then the result follows immediately by the fact that $f \in C_0\left(W^N\right)$ and the dominated convergence theorem. Now, assume $x_N^{(n)}$ remains bounded. We first take for each n a sequence $\{x^{(n),m}\}_m \in \mathring{W}^{N+1}$ such that $\lim_{m\to\infty} x^{(n),m} = x^{(n)}$. For $z \in \mathring{W}^{N+1}$ we have using the explicit expression:

$$\left[\Lambda_N^{N+1} f\right](z_1, \cdots, z_{N+1}) = \frac{N! \int_{z_1}^{z_2} \cdots \int_{z_N}^{z_{N+1}} \Delta_N(y) f(y) dy_1 \cdots dy_N}{\Delta_{N+1}(z)}. \tag{3.3}$$

Applying the Mean Value Theorem, a total of N times, successively in the variables $z_{N+1}, z_N, \cdots, z_2$ to the function:

$$F_{N+1}(z_1, \cdots, z_{N+1}) = \int_{z_1}^{z_2} \int_{z_2}^{z_3} \cdots \int_{z_N}^{z_{N+1}} \Delta_N(y) f(y) dy_1 \cdots dy_N$$

we obtain:

$$\left[\Lambda_N^{N+1} f\right](z_1, \cdots, z_{N+1}) = \frac{N! \prod_{i=1}^N (z_{i+1} - z_i) \Delta_N(\xi) f(\xi)}{\Delta_{N+1}(z)} \tag{3.4}$$

for some $(\xi_1, \cdots, \xi_N)$ such that $z_1 < \xi_1 < z_2 < \cdots < \xi_N < z_{N+1}$. Moreover, note that the interlacing constraints for $i = 1, \cdots, N-1$ and $l = 1, \cdots, N-i$ imply:

$$\frac{|\xi_{i+l} - \xi_i|}{|z_{i+l+1} - z_i|} \le 1. \tag{3.5}$$

Then, we have:

$$\begin{aligned}\left[\Lambda_N^{N+1} f\right](x_1^{(n)}, \cdots, x_{N+1}^{(n)}) &= \lim_{m\to\infty} \left[\Lambda_N^{N+1} f\right](x_1^{(n),m}, \cdots, x_{N+1}^{(n),m}) \\ &= \lim_{m\to\infty} \frac{N! \prod_{i=1}^N (x_{i+1}^{(n),m} - x_i^{(n),m}) \Delta_N(\xi^{(n),m}) f(\xi^{(n),m})}{\Delta_{N+1}(x^{(n),m})} \\ &= \lim_{m\to\infty} \frac{N! \prod_{i=1}^N (x_{i+1}^{(n),m} - x_i^{(n),m}) \Delta_N(\xi^{(n),m})}{\Delta_{N+1}(x^{(n),m})} f\left(\xi^{(n)}\right).\end{aligned}$$

If $\xi_N^{(n)} \to \infty$, then $f\left(\xi^{(n)}\right) \to 0$ and moreover since we have uniformly in n and m the bound from the constraints (3.5):

$$\left|\frac{\prod_{i=1}^N (x_{i+1}^{(n),m} - x_i^{(n),m}) \Delta_N(\xi^{(n),m})}{\Delta_{N+1}(x^{(n),m})}\right| \le 1,$$

we get $\left[\Lambda_N^{N+1} f\right](x_1^{(n)}, \cdots, x_{N+1}^{(n)}) \to 0$. Now, suppose $\xi_N^{(n)}$ remains bounded. Note that, we have:

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{\left(x_{N+1}^{(n),m} - x_{N-1}^{(n),m}\right)} = 0$$

since $x_{N-1}^{(n)} \le x_N^{(n)}$ remains bounded. While on the other hand, since $\xi_N^{(n)}$ is bounded, we have that:

$$\left|\frac{\left(x_{N+1}^{(n),m} - x_{N-1}^{(n),m}\right) \prod_{i=1}^N (x_{i+1}^{(n),m} - x_i^{(n),m}) \Delta_N(\xi^{(n),m}) f\left(\xi^{(n)}\right)}{\Delta_{N+1}(x^{(n),m})}\right|$$

remains bounded from which the result follows.

□

We will now consider the projective limit of this system $(W^N, \Lambda_N^{N+1})_{N\ge1}$ in the measurable category $\mathcal{B}$. Namely $\mathcal{B}$ consists of objects given by standard Borel spaces and morphisms given by Markov kernels that we will also call links. Such a kernel $\Lambda : X \to Y$ between two standard Borel spaces X and Y is a function $\Lambda(x, \mathcal{Y})$ where x ranges over X and $\mathcal{Y}$ ranges over measurable subsets of Y such that $\Lambda(x, \cdot)$ is a probability measure on Y for any fixed $x \in X$ and $\Lambda(\cdot, \mathcal{Y})$ is a measurable function on X for each fixed $\mathcal{Y}$. A limit object W^∞ is understood in the following sense:

It consists of an object $W^\infty = \varprojlim W^N$ and links $\Lambda_N^\infty : W^\infty \to W^N$ such that $\Lambda_N^\infty \Lambda_K^N = \Lambda_K^\infty$, $\forall K < N$. Moreover if an object $\tilde{W}^\infty$ and links $\tilde{\Lambda}_N^\infty : \tilde{W}^\infty \to W^N$ satisfy the same condition, then there exists a unique link $\Lambda^{\tilde{W}^\infty}_{W^\infty} : \tilde{W}^\infty \to W^\infty$ such that $\tilde{\Lambda}^\infty_N = \Lambda^{\tilde{W}^\infty}_{W^\infty}\Lambda^\infty_N$. By a general result of Winkler, see Theorem 4.1.3 in [168], the limit exists and it is unique up to a Borel isomorphism (more generally this fact holds for arbitrary standard Borel spaces in place of the Weyl chambers W^N). We will call W^∞ the boundary of the system $(W^N, \Lambda^{N+1}_N)_{N\geq 1}$.

In fact the boundary coincides with the following construction: Note that the links induce the chain of affine mappings:

$$\cdots \to \mathcal{M}_p\left(W^{N+1}\right) \to \mathcal{M}_p\left(W^N\right) \to \cdots \to \mathcal{M}_p\left(W^2\right) \to \mathcal{M}_p\left(W^1\right),$$

where $\mathcal{M}_p\left(W^N\right)$ is the simplex of probability measures on W^N topologised with the weak topology. Consider the space $\mathcal{W} = \prod_{N=1}^\infty \mathcal{M}_p\left(W^N\right)$ with the product topology and define the inverse system of simplices (not to be confused with the limit in the measurable category):

$$\lim_{\leftarrow}\mathcal{M}_p\left(W^N\right) = \{(\mu_N)_{N\geq 1} \in \mathcal{W} : \mu_{N+1}\Lambda^{N+1}_N = \mu_N \text{ , } \forall N\},$$

consisting of coherent sequences of measures. By Theorem 3.2.3 in [168] (see also step 3 in the proof of Theorem 4.1.3 therein) the convex set $\lim_{\leftarrow}\mathcal{M}_p\left(W^N\right)$ is actually a Polish simplex. Moreover, by steps 3 and 4 in the proof of Theorem 4.1.3 on page 103 of [168] (see also second paragraph on page 109 of [168]) its extreme points coincide with W^∞ (in fact this is how W^∞ is constructed):

$$W^\infty = \lim_{\leftarrow} W^N = \mathrm{Ex}\left(\lim_{\leftarrow}\mathcal{M}_p\left(W^N\right)\right). \tag{3.6}$$

Thus, the boundary consists of extremal coherent sequences of (probability) measures. Moreover, for $\mathfrak{w} \in W^\infty$ such that $\mathfrak{w} = (\mu_N)_{N\geq 1} \in \mathrm{Ex}\left(\lim_{\leftarrow}\mathcal{M}_p\left(W^N\right)\right)$ the links are given by $\Lambda^\infty_N(\mathfrak{w}, \cdot) = \mu_N(\cdot)$. Finally, if all the links $\{\Lambda^{N+1}_N\}_{N\geq 1}$ and $\{\Lambda^\infty_N\}_{N\geq 1}$ are Feller we will say that W^∞ is the Feller boundary of $\{W^N\}_{N\geq 1}$.Then, we have the following proposition (proven in this subsection after several preliminaries),

Proposition 3.4. *$W^\infty = \Omega$ is the Feller boundary of $\{W^N\}_{N\geq 1}$.*

We start by recalling the following crucial observation originally made (in published form) by Borodin and Olshanski in [25] (see graph of spectra remarks pages 30-31 of [25]). Let $\mathfrak{M}$ be any $\mathbb{U}(\infty)$ invariant probability measure on H and let $\mu^{\mathfrak{M}}_N = \left(\mathsf{eval}_N \circ \pi^\infty_N\right)_* \mathfrak{M}$ be the (ordered) radial part of the projection $(\pi^\infty_N)_*(\mathfrak{M})$ of $\mathfrak{M}$ on $H(N)$, i.e a measure on W^N. Then, $\forall N \geq 1$,

$$\mu^{\mathfrak{M}}_{N+1}\Lambda^{N+1}_N = \mu^{\mathfrak{M}}_N.$$

Conversely, any coherent sequence of probability measures $\{\mu_N\}_{N\geq 1}$ comes from a $\mathbb{U}(\infty)$ invariant measure $\mathfrak{M}$. A proof of these statements immediately follows also from Lemma 3.3 and Lemma 3.8 of [53] for example (see also Proposition 3.1 of [121]). Thus, there exists a bijection between coherent measures and $\mathbb{U}(\infty)$ invariant probability measures on H. More formally, we have that $\mathcal{M}_p(\Omega) = \mathcal{M}_p\left(\mathcal{M}_p^{\mathbb{U}(\infty)-erg}(H)\right) = \mathcal{M}_p^{\mathbb{U}(\infty)-inv}(H)$. With this identification consider the map between convex sets $\Phi : \mathcal{M}_p(\Omega) \to \varprojlim \mathcal{M}_p(W^N)$:

$$\Phi(\mathfrak{M}) = \left\{ \left(\mathsf{eval}_N \circ \pi_N^\infty\right)_* \mathfrak{M} \right\}_{N\geq 1},$$

which is an affine bijection and hence we have the following lemma (the reader obviously notices that all the hard work is transferred from Theorem 3.1, which we are essentially reinterpreting following [25]),

Lemma 3.5. *We have a bijection between* Ω *and* $\operatorname{Ex}\left(\varprojlim \mathcal{M}_p\left(W^N\right)\right)$.

We make this more explicit and we begin by defining the following Markov kernels Λ_N^∞ from Ω to W^N for $\omega \in \Omega$ with $\gamma_2(\omega) > 0$,

$$\Lambda_N^\infty(\omega, dx) = \left(\prod_{k=1}^{N-1} \frac{1}{k!}\right) \det\left(\phi_\omega^{(j-1)}(x_{N+1-i})\right)_{i,j=1}^N \Delta_N(x) dx, \tag{3.7}$$

from Ω to W^N where ϕ_ω as before is such that $\hat{\phi}_\omega(\xi) = F_\omega(\xi)$. Obviously, $\Lambda_N^\infty(\cdot, dx)$ is measurable on Ω. Moreover, the positivity property, $\Lambda_N^\infty(\omega, dx) \geq 0$, immediately follows from the fact that ϕ_ω is extended totally positive. To obtain the following coherency relation

$$\Lambda_{N+1}^\infty \Lambda_N^{N+1} = \Lambda_N^\infty,$$

observe that,

$$\begin{aligned}
\Lambda_{N+1}^\infty \Lambda_N^{N+1}(\omega, dy) &= \left(\prod_{k=1}^{N-1} \frac{1}{k!}\right) \Delta_N(y) \int_{-\infty}^{y_1} \cdots \int_{y_N}^{\infty} \det\left(\phi_\omega^{(j-1)}(x_{N+2-i})\right)_{i,j=1}^{N+1} dx_1 \cdots dx_{N+1} dy \\
&= \left(\prod_{k=1}^{N-1} \frac{1}{k!}\right) \Delta_N(y) \det\left(\phi_\omega^{(j-1)}(y_{N+1-i})\right)_{i,j=1}^{N} dy.
\end{aligned}$$

To see this first note that the integral is equal to:

$$\det\begin{bmatrix}
\int_{y_N}^\infty \phi_\omega(x_{N+1}) dx_{N+1} & \cdots & -\phi_\omega^{(N-1)}(y_N) \\
\vdots & \ddots & \vdots \\
\int_{y_1}^{y_2} \phi_\omega(x_2) dx_2 & \cdots & \phi_\omega^{(N-1)}(y_2) - \phi_\omega^{(N-1)}(y_1) \\
\int_{-\infty}^{y_1} \phi_\omega(x_1) dx_1 & \cdots & \phi_\omega^{(N-1)}(y_1)
\end{bmatrix}_{(N+1)\times(N+1)}.$$

Now successively add row i to row $i-1$, starting from $i = N+1$. The identity then follows from the fact that the first row has a 1 as its first entry since $\int_{-\infty}^\infty \phi_\omega(x) dx = 1$ and 0's elsewhere. Finally, to see that Λ_N^∞ is correctly normalized, i.e. $\Lambda_N^\infty 1 = 1$, observe that from the coherency relation $\Lambda_N^\infty \Lambda_1^N = \Lambda_1^\infty$ and the facts that $\Lambda_1^N 1 = 1$ and $\Lambda_1^\infty 1 = 1$ this follows immediately.

We now extend the definition to arbitrary $\omega \in \Omega$. We first observe, that in fact for ω with $\gamma_2(\omega) > 0$ if we consider the measure $M_\omega(dX)$ on H with characteristic function F_ω as in Theorem 3.1, then (see proof of Theorem 7.7 of [123]) $\Lambda_N^\infty(\omega, dx)$ is the radial part of its projection on $H(N)$, more formally $\Lambda_N^\infty(\omega, dx) = \left(\mathsf{eval}_N \circ \pi_N^\infty\right)_* M_\omega(dx)$. In particular, for any $\omega \in \Omega$ we can define $\Lambda_N^\infty(\omega, dx)$ as the radial part of the projection of M_ω on $H(N)$ or equivalently as the unique weak limit, this essentially follows from Levy's continuity theorem and will also be detailed in Lemma 3.7 below, as $\omega_{\gamma_2}(n) \to \omega$ (where $\{\omega_{\gamma_2}(n)\}_n$ is any sequence in Ω such that $\gamma_2\left(\omega_{\gamma_2}(n)\right) > 0$ and $\omega_{\gamma_2}(n) \to \omega$) of $\Lambda_N^\infty(\omega_{\gamma_2}(n), dx)$ namely,

$$\Lambda_N^\infty(\omega_{\gamma_2}(n), dx) \rightharpoonup \Lambda_N^\infty(\omega, dx), \text{ as } n \to \infty.$$

Hence, we have obtained the following lemma,

Lemma 3.6. *For all $\omega \in \Omega, N \geq 1$ the kernels $\Lambda_N^\infty(\omega, dx)$ are Markov and satisfy,*

$$\Lambda_{N+1}^\infty \Lambda_N^{N+1} = \Lambda_N^\infty.$$

Note that, see Remark 3.2, for $\gamma_1(\omega), \alpha_i^+(\omega), \alpha_i^-(\omega) = 0$ then $\Lambda_N^\infty(\omega, dx)$ is just the N-particle *GUE* with variance γ_2. Moreover, for $\gamma_2(\omega), \alpha_i^+(\omega), \alpha_i^-(\omega) = 0$ then $\Lambda_N^\infty(\omega, dx)$ is the delta measure on the scalar matrix $\gamma_1(\omega)\mathrm{Id}_N$, in particular $\Lambda_N^\infty(\omega, dx)$ need not have a smooth density with respect to Lebesgue measure. As already anticipated, these kernels are Feller,

Lemma 3.7. *The kernels $\{\Lambda_N^\infty\}_{N\geq 1}$ are Feller.*

Proof. We want to show that $(\Lambda_N^\infty f)(\omega) \in C_0(\Omega)$ whenever $f \in C_0(W^N)$. By the density of the Schwartz functions $\mathcal{S}(W^N)$ (smooth with all derivatives decreasing faster than any inverse power of x as $x \to \pm\infty$) in $C_0(W^N)$ it suffices to check this for $f \in \mathcal{S}(W^N)$. The following equality, which is a multidimensional version of the usual Plancherel theorem, is the key tool. It is also the main content of the proof of Theorem 7.7 of Olshanski and Vershik [123] and is the equality of displays 7.10 and 7.18 therein. For ω with $\gamma_2(\omega) > 0$,

$$const \times \int_{W^N} \det\left(\phi_\omega^{(j-1)}(x_{N+1-i})\right)_{i,j=1}^N \Delta_N(x) f(x) dx = \int_{\mathbb{R}^N} \Delta_N^2(x) F_\omega(x_1) \cdots F_\omega(x_N) \bar{\hat{f}}(x) dx,$$

where *const* is a positive constant whose exact value will not be important in what follows. Thus, by going to Fourier space we can relate $(\Lambda_N^\infty f)(\omega)$ to the functions F_ω for which we have explicit expressions,

$$(\Lambda_N^\infty f)(\omega) = Const \times \int_{\mathbb{R}^N} \Delta_N^2(x) F_\omega(x_1) \cdots F_\omega(x_N) \bar{\hat{f}}(x) dx. \tag{3.8}$$

Furthermore, recall that the Fourier transform $\hat{f}$ of $f \in \mathcal{S}(W^N)$ is still in $\mathcal{S}(W^N)$. Now, observe that (3.8) makes sense for arbitrary ω, even with $\gamma_2(\omega) = 0$. Similarly, in order to show continuity in general, first suppose $\omega_n \to \omega$ then, since for any fixed $x \in \mathbb{R}$, $F_\omega(x)$ as a function of $\omega \in \Omega$ is continuous:

$$\Delta_N^2(x)F_{\omega_n}(x_1)\cdots F_{\omega_n}(x_N)\bar{\hat{f}}(x) \to \Delta_N^2(x)F_{\omega}(x_1)\cdots F_{\omega}(x_N)\bar{\hat{f}}(x) \;\; a.e. \,,$$

and thus, by the dominated convergence theorem,

$$(\Lambda_N^\infty f)(\omega_n) \to (\Lambda_N^\infty f)(\omega).$$

Now, in order to show that $(\Lambda_N^\infty f)(\omega)$ vanishes as $\omega \to \infty$ we note that $\omega \to \infty$ is actually equivalent to any combination of the following cases, $\gamma_1 \to \pm\infty$ or $\gamma_2 \to \infty$ or $\alpha_1^\pm \to \infty$. Observe that any of these possibilities can occur on its own. First, suppose that $\gamma_2 \to \infty$. We see that, since $\hat{f} \in \mathcal{S}(W^N)$, there exists $R < \infty$ such that,

$$\int_{x \notin [-R,R]^N} |\Delta_N^2(x)F_\omega(x_1)\cdots F_\omega(x_N)\bar{\hat{f}}(x)| dx \lesssim \epsilon.$$

And thus,

$$(\Lambda_N^\infty f)(\omega) \lesssim \epsilon + \int_{[-R,R]^N} \left|\Delta_N^2(x)F_\omega(x_1)\cdots F_\omega(x_N)\bar{\hat{f}}(x)\right| dx \lesssim \epsilon + \left(\int_{-R}^{R} \left|F_\omega(y)\right| dy\right)^N.$$

But we have,

$$\left|F_\omega(y)\right| \le e^{-\frac{\gamma_2}{2}y^2} \;\text{ in }\; [-R,R],$$

and so $\left|F_\omega(y)\right| \to 0$ as $\gamma_2 \to \infty$ $\forall y \in [-R,R]\backslash\{0\}$ and $|F_\omega(0)| = 1$ (in particular bounded). Hence, using the dominated convergence theorem we obtain,

$$\left(\int_{-R}^{R} \left|F_\omega(y)\right| dy\right)^N \to 0 \;\text{ as }\; \gamma_2 \to \infty.$$

Of course the integral above can be explicitly calculated in terms of the error function from which the result is evident as well. Now, in order to show that $(\Lambda_N^\infty f)(\omega)$ vanishes as $\alpha_1^\pm \to \infty$ we follow the same argument, except that now we use the bound,

$$\left|F_\omega(y)\right| \le \frac{1}{\sqrt{1+\left(\alpha_1^+ y\right)^2}} \frac{1}{\sqrt{1+\left(\alpha_1^- y\right)^2}} \;\text{ in }\; [-R,R],$$

and thus $\left|F_\omega(y)\right| \to 0$ as either $\alpha_1^\pm \to \infty$, $\forall y \in [-R,R]\backslash\{0\}$ from which the claim follows. We finally assume that $\gamma_1 \to \pm\infty$ and take a different approach. First, we write $\Lambda_N^\infty f$ as follows, viewing it as a function of γ_1,

$$(\Lambda_N^\infty f)(\gamma_1) = Const \times \int_{\mathbb{R}^N} e^{i\gamma_1 x_1+\cdots+i\gamma_1 x_N}\left(\prod_{j=1}^N e^{-\frac{\gamma_2}{2}x_j^2}\right)\prod_{k=1}^\infty\left(\prod_{j=1}^N \frac{e^{-i\alpha_k^+ x_j}}{1-i\alpha_k^+ x_j}\right)\prod_{k=1}^\infty\left(\prod_{j=1}^N \frac{e^{i\alpha_k^- x_j}}{1+i\alpha_k^- x_j}\right)\left(\Delta_N^2(x)\bar{\hat{f}}(x)\right)dx$$

and noting that this is exactly *Fourier inversion* of a product which is given in terms of a convolution up to some numerical constant $\tilde{C}$ as follows,

$$(\Lambda_N^\infty f)(\gamma_1) = \tilde{C} \times \left(\phi_{\gamma_2}^{\otimes N} * \phi_{\alpha_1^+}^{\otimes N} * \cdots * \phi_{\alpha_1^-}^{\otimes N} * \cdots * g\right)(\gamma_1, \cdots, \gamma_1),$$

where $g \in \mathcal{S}(W^N)$ is such that $\hat{g}(\xi) = \Delta_N^2(\xi)\bar{\hat{f}}(\xi)$ .The fact that $(\Lambda_N^\infty f)(\gamma_1) \to 0$, as $\gamma_1 \to \pm\infty$ now follows, since it is a convolution of $L^1(\mathbb{R}^N)$ functions (in fact it is a Schwartz function). We finally remark that the argument above is essentially just the Riemann-Lebesgue lemma. □

We are finally ready to provide a full proof of Proposition 3.4,

Proof of Proposition 3.4. By making use of Lemmas 3.5, 3.6 and 3.7 we get that the map $\Lambda^\infty : \Omega \to \mathrm{Ex}\left(\varprojlim \mathcal{M}_p\left(W^N\right)\right)$ is a continuous (part of the statement of Lemma 3.7) bijection. We obtain that it is actually a Borel isomorphism by Theorem 3.2 in [104], which states that a Borel one to one map from a standard Borel space onto a subset of a countably generated Borel space is a Borel isomorphism or in this particular setting see Proposition 9.4 of [25]. This extends to a Borel isomorphism between $\mathcal{M}_p(\Omega)$ and $\varprojlim \mathcal{M}_p\left(W^N\right)$ by making use of Theorem 9.1 of [25] (or more generally the ergodic decomposition theorem for actions of inductively compact groups of Bufetov, namely Theorem 1 in [42], see also the proof of Theorem 4.1.3 of [168]). Finally, the Feller assertion follows from Lemmas 3.3 and 3.7. □

3.2.3 Markov Processes on the boundary

We now, briefly recall the Borodin Olshanski formalism (see in particular Section 2 of [28] for detailed proofs), the so called *method of intertwiners*, for constructing Markov processes on the boundary Ω (Ω could in more generality be any locally compact metrizable topological space with a countable base which arises as the *Feller* boundary of some projective sequence $\{E_N\}_{N\geq 1}$ in the sense described above).

Hence, let $\{P_N(t)\}_{N\geq 1}$ be a sequence of Markov semigroups on W^N consistent with the *Feller* links above namely,

$$P_{N+1}(t)\Lambda_N^{N+1} = \Lambda_N^{N+1} P_N(t) \text{ , } \forall t \geq 0, \ \forall N \geq 1.$$

Then, we have the following Theorem, proven as Proposition 2.4 in [28] (or more precisely a special case of that result applied to our situation),

Theorem 3.8. *There exists a unique Markov semigroup $P_\infty(t)$ on Ω so that $\forall N \geq 1$ we have $\forall t \geq 0$,*

$$P_\infty(t)\Lambda_N^\infty = \Lambda_N^\infty P_N(t).$$

Moreover, in case the semigroups $P_N(t)$ are Feller then so is $P_\infty(t)$.

Invariant measures It can be easily seen that, if $\forall N \geq 1$, μ_N is an invariant measure of $P_N(t)$ and these measures are compatible with the links then the measure μ on Ω given by,

$$\mu \Lambda_N^{\infty} = \mu_N,$$

is invariant for $P_\infty(t)$. If furthermore, we assume that, $\forall N \geq 1$ μ_N is the *unique* invariant measure of $P_N(t)$ (in such case, compatibility with the links is immediate) then μ is the *unique* invariant measure for $P_\infty(t)$.

3.3 Hua-Pickrell measures

In this brief section we define the Hua-Pickrell measures, depending on a complex parameter s. As already mentioned in the introduction, these measures were first studied for *real* s by Hua in [81] and then much later by Pickrell in [128], unaware of Hua's earlier work. The possibility of the parameter s being complex was first investigated by Neretin in [108].

We will assume throughout that $\Re(s) > -\frac{1}{2}$. This restriction is necessary in order for the measures to be finite. In particular, we assume that all of them are normalized to have mass 1. In recent years, also infinite Hua-Pickrell measures (with $\Re(s) \leq -\frac{1}{2}$) have been intensively studied by Bufetov and Qiu, see for example [43] and the references therein. We consider the following measures on $\mathbb{U}(N)$ given by,

$$const \times \det\left((I+U)^{\bar{s}}\right)\det\left((I+U^{-1})^{s}\right) \times dU,$$

where dU denotes Haar measure on $\mathbb{U}(N)$. We note that, for $s = 0$, this is just the Circular Unitary Ensemble (CUE). The projection of this measure on the eigenvalues $(u_1, \cdots, u_N)$ or equivalently the eigenangles, with $u_j = e^{i\theta_j}$ is given by,

$$const \times \prod_{1\leq j<k\leq N} |u_j - u_k|^2 \prod_{j=1}^{N} (1+u_j)^{\bar{s}}(1+\bar{u_j})^{s} \times d\theta_j.$$

Under the inverse Cayley transform $\mathfrak{C}^{-1}$ the corresponding measure on $H(N)$ denoted by $\mathsf{M}_{HP}^{s,N}$ becomes,

$$\mathsf{M}_{HP}^{s,N}(dX) = const \times \det\left((I+iX)^{-s-N}\right)\det\left((I-iX)^{-\bar{s}-N}\right) \times dX, \tag{3.9}$$

where dX denotes Lebesgue measure on $H(N)$. Looking at the radial part of $\mathsf{M}_{HP}^{s,N}(dX)$ we get a probability measure on W^N which we will denote by $\mu_{HP}^{s,N}$ and will be referring to as a

Hua-Pickrell measure and which is given by,

$$\mu_{HP}^{s,N}(dx) = const \times \Delta_N^2(x)\prod_{j=1}^{N}(1+ix_j)^{-s-N}(1-ix_j)^{-\bar{s}-N}dx_j$$
$$= const \times \Delta_N^2(x)\prod_{j=1}^{N}(1+x_j^2)^{-\Re(s)-N}e^{2\Im(s)Arg(1+ix_j)}dx_j.$$

A remarkable property of these measures is that they are coherent with the respect to the links, see for example Proposition 3.1 of [25] for a direct proof,

$$\mu_{HP}^{s,N+1}\Lambda_N^{N+1} = \mu_{HP}^{s,N}.$$

This statement will also be derived as Corollary 3.17 as a consequence of our intertwining relations between Markov semigroups. We finally denote by μ_{HP}^{s} the corresponding measure on Ω. It can be easily seen, that the measures $\mu_{HP}^{s,N}$, $\forall N \geq 1$ give rise to determinantal point processes. By an approximation procedure, μ_{HP}^{s} does so as well, and this was the main objective of the study of [25].

Remark 3.9. *In fact, the situation is a bit more subtle, μ_{HP}^{s} gives rise to a determinantal point process in $\mathbb{R}^*$ where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ under the so called forgetting map that disregards $\gamma_1(\omega)$ and $\gamma_2(\omega)$ and $\alpha_i^+(\omega), \alpha_j^-(\omega)$ that are zero namely,*

$$\omega = \left(\{\alpha_i^+(\omega)\}, \{\alpha_i^+(\omega)\}, \gamma_1(\omega), \gamma_2(\omega)\right) \mapsto \left(-\alpha_1^-(\omega), -\alpha_2^-(\omega), \cdots, \alpha_2^+(\omega), \alpha_1^+(\omega)\right) \in Conf(\mathbb{R}^*).$$

However, in a recent breakthrough Qiu in [130], has proven that for $s \in \mathbb{R}$ (this covers both the finite and infinite cases) the measure μ_{HP}^{s} only charges the subset $\Omega_0 \subset \Omega$ such that for $\omega \in \Omega_0$,

$$\alpha_i^+(\omega) \neq 0, \alpha_j^-(\omega) \neq 0, \gamma_2(\omega) = 0 \text{ and } \gamma_1(\omega) = \lim_{n\to\infty}\left(\sum_{l\in\mathbb{Z}^*} x_l(\omega)\mathbf{1}_{|x_l(\omega)|>\frac{1}{n^2}}\right)$$

where,

$$x_l(\omega) = \begin{cases}\alpha_l^+(\omega) \text{ if } l > 0\\ -\alpha_l^-(\omega) \text{ if } l < 0\end{cases}.$$

Remark 3.10. *For $s = 0$, under the forgetting map above and the transform $x \mapsto y = -\frac{1}{\pi x}$ the measure μ_{HP}^{0} gives rise to the sine point process, abbreviated Sine_2 here, that is the determinantal point process on $\mathbb{R}$ with correlation kernel given by,*

$$K_{\mathsf{Sine}_2}(x,y) = \frac{\sin(\pi(y-x))}{\pi(y-x)},$$

(see Theorem I of [25]). In particular, the dynamics obtained in Corollary 3.18 below, under this

transform will leave the Sine_2 *process invariant.*

3.4 Hua-Pickrell Diffusions

Before proceeding to define our stochastic dynamics, we remark in passing that, all our dynamical results are valid for any $s \in \mathbb{C}$ and not just for $\Re(s) > -\frac{1}{2}$. So, we begin by considering the one dimensional diffusions that will constitute our basic building blocks. These are strong Markov processes, with continuous sample paths in $\mathbb{R}$, with both $-\infty$ and ∞ as natural boundaries and generators given by,

$$L_s^{(n)} = (x^2+1)\frac{d^2}{dx^2} + \left[(2-2n-2\Re(s))\,x + 2\Im(s)\right]\frac{d}{dx},$$

with invariant/speed measure with density with respect to Lebesgue measure given by,

$$m_s^{(n)}(x) = (1+x^2)^{-\Re(s)-n}e^{2\Im(s)Arg(1+ix)},$$

and (the non-exploding) *SDE* description,

$$dX(t) = \sqrt{2(X^2(t)+1)}dW(t) + \left[(2-2n-2\Re(s))\,X(t) + 2\Im(s)\right]dt.$$

We will denote by $p_t^{(n),s}(x,y)$ its transition density in $\mathbb{R}$ with respect to Lebesgue measure.

Moving on, we note that $\Delta_n(x)$ is a positive eigenfunction of n copies of $L_s^{(n)}$-diffusions with eigenvalue denoted by $\lambda_{n,s}$. More precisely,

Lemma 3.11. *We have* $\sum_{i=1}^n L_{s,x_i}^{(n)}\Delta_n(x) = \lambda_{n,s}\Delta_n(x)$ *where* $\lambda_{n,s} = \frac{n(n-1)(-2n+1-3\Re(s))}{3}$.

Proof. First, observe that the operator $\sum_{i=1}^n L_{s,x_i}^{(n)}$ is symmetric and when applied to a polynomial does not raise the degree. Thus, $\sum_{i=1}^n L_{s,x_i}^{(n)}\Delta_n(x)$ is antisymmetric, divisible by $\Delta_n(x)$ and of the same degree and so actually a multiple of $\Delta_n(x)$. Finally, the coefficient of $x_n^{n-1}x_{n-1}^{n-2}\cdots x_2$ after the application of $\sum_{i=1}^n L_{s,x_i}^{(n)}$ gives $\lambda_{n,s}$. The lemma can also be obtained by iteration of the intertwining relations of the next section. □

As in the introduction, we denote by $P_{HP}^{s,N}(t)$ the Karlin-McGregor semigroup of N $L_s^{(N)}$-diffusions h-transformed by $\Delta_N(x)$, namely the semigroup having kernel with (t,x,y) in $(0,\infty)\times \mathring{W}^N \times W^N$ given by,

$$e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)}\det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N dy.$$

The Markov process associated to it, is equivalently given by the unique strong solution, as we see in Lemma 3.12 below, of the system of *SDEs*,

$$dX_i(t) = \sqrt{2(X_i^2(t)+1)}dW_i(t) + \left[(2-2N-2\Re(s))\,X_i(t) + 2\Im(s) + \sum_{j\neq i}\frac{2(X_i^2(t)+1)}{X_i(t)-X_j(t)}\right]dt, \quad (3.10)$$

where the $\{W_i\}_{i=1}^N$ are independent standard Brownian motions.

Lemma 3.12. *The system of SDEs (3.10) has a unique strong solution. Moreover, its transition semigroup is given by $P_{HP}^{s,N}(t)$.*

Proof. We first prove that the system of *SDEs* (3.10) has a unique strong solution with no collisions and no explosions, even if started from a "degenerate" point (when some of the coordinates coincide i.e. there is instant "diffraction" of particles). This follows by applying Theorem 2.2 of [78] whose conditions we now proceed to check. In order to apply the aforementioned Theorem, one needs to note that we can write,

$$\sum_{j\neq i}\left(\frac{2(1+x_ix_j)}{x_i-x_j}\right)+(2N-2)x_i=\sum_{j\neq i}\left(\frac{2(1+x_i^2)}{x_i-x_j}\right),$$

and thus, one can identify the function $H:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ in Theorem 2.2 of [78] as follows,

$$H(x,y)=2(1+xy).$$

The conditions (A1)-(A5),(C1),(C2) on $\sigma(x)=\sqrt{2(1+x^2)}, b(x)=2\Im(s)-2\Re(s)x, H(x,y)=2(1+xy)$ can then be checked as follows. First of all, (C1) and (C2) correspond to a Yamada-Watanabe and growth conditions at $\pm\infty$ respectively for σ and b which are immediate. Now, condition (A1) requires that for $w<x<y<z$,

$$\frac{1+wz}{z-w}\leq\frac{1+xy}{y-x}.$$

To see this, define for fixed $w<x<y$ the LHS to be $f(z)=\frac{1+wz}{z-w}$. Since,

$$\frac{d}{dz}f(z)=-\frac{1+w^2}{(z-w)^2},$$

and for $z=y$ the inequality $\frac{1+wy}{y-w}\leq\frac{1+xy}{y-x}$ is equivalent to $(x-w)(1+y^2)\geq 0$, the statement is immediately seen to be true. Moving on to (A2), we require the existence of a constant $c\geq 0$ such that,

$$\sigma^2(x)+\sigma^2(y)\leq c(x-y)^2+4H(x,y).$$

Any choice of $c\geq 3$ will do, since,

$$x^2+y^2+2xy+4\geq 0.$$

For condition (A3), we need to find a constant $c\geq 0$ such for $x<y<z$,

$$(1+xy)(y-x)+(1+yz)(z-y)\leq c(z-y)(z-x)(y-x)+(1+xz)(z-x).$$

Defining $g_c(y)$, for fixed $x < z$ by,

$$g_c(y) = c(z-y)(z-x)(y-x) + (1+xz)(z-x) - (1+xy)(y-x) - (1+yz)(z-y),$$

we see that this is a quadratic function in y with zeros at $y = x$ and $y = z$ and leading coefficient $(z-x)(1-c)$. Thus, for $c > 1$ we see that $g_c(y) \geq 0$ in $[x,z]$ and the statement follows. Condition (A4) obviously holds, since $\sigma^2(x) + H(x,x) > 0 \ \forall x \in \mathbb{R}$ and finally since $\sigma(x), b(x)$ do not depend on i, condition (A5) is vacuous. Hence, the law of $((X_1(t), \cdots, X_N(t)); t \geq 0)$ from (3.10) is the unique solution to the well-posed martingale problem with generator acting on $C_c^2\left(W^N(\mathbb{R})\right)$ (twice continuously differentiable functions with compact support in $W^N(\mathbb{R})$) given by,

$$\mathsf{L}_s^{(N)} = \Delta_N^{-1}(x) \circ \left(\sum_{i=1}^N L_{s,x_i}^{(N)}\right) \circ \Delta_N(x) - \lambda_{N,s}.$$

We can now easily observe that, this is exactly given by a Doob's h-transform of N one dimensional diffusions; with transition kernel having density with respect to Lebesgue measure in $(0,\infty) \times \mathring{W}^N \times W^N$ given by,

$$e^{-\lambda_{N,s}t} \frac{\Delta_N(y)}{\Delta_N(x)} \det\left(p_t^{(N),s}(x_i, y_j)\right)_{i,j=1}^N,$$

where $p_t^{(N),s}(x,y)$ is the *Feller* transition density of the one dimensional diffusion process with generator $L_s^{(N)}$ with two natural boundaries. □

We now give a direct and rather technical proof that the semigroups are Feller. A much neater argument is given in section 3.8, however one needs to introduce a quite non-trivial matrix valued stochastic process, having (3.10) as its eigenvalue evolution. The matrix process is in some sense better behaved from an *SDE* point of view, so we can appeal to existing results in the literature.

Lemma 3.13. *The semigroups $P_{HP}^{s,N}(t)$ are Feller in the sense that $\forall f \in C_0(W^N)$ we have,*

$$P_{HP}^{s,N}(t) f \in C_0(W^N), \quad \forall t > 0,$$
$$\lim_{t \to 0} P_{HP}^{s,N}(t) f = f.$$

Proof. For each $(x_1 \leq \cdots \leq x_N) \in W^N$ the continuity of $t \mapsto \mathbb{E}_{(x_1,\cdots,x_N)}\left[f\left(X_1(t), \cdots, X_N(t)\right)\right]$ with $f \in C_0$ follows from the fact that $(X_1(t), \cdots, X_N(t); t \geq 0)$ is the unique strong solution of the system of *SDEs* even if started from the diagonals. More specifically, this follows by the almost sure continuity in t of $(X_1(t), \cdots X_N(t); t \geq 0)$ (see statement of Theorem 5.1 of [78]).

For the fact that $P_{HP}^{s,N}(t) f \in C_0$ if $f \in C_0$, first pick R such that $|f(x_1, \cdots, x_N)| \leq \epsilon$ for

$(x_1 \le \cdots \le x_N) \notin [-R,R]^N$ and let us write for $(x_1,\cdots,x_N) \in W^N$,

$$\begin{aligned}|\mathbb{E}_{(x_1,\cdots,x_N)}\left[f\left(X_1(t),\cdots,X_N(t)\right)\right]| \le \mathbb{E}_{(x_1,\cdots,x_N)}\left[|f\left(X_1(t),\cdots,X_N(t)\right)|\mathbf{1}\left(X(t)\in[-R,R]^N\right)\right]+\\ \mathbb{E}_{(x_1,\cdots,x_N)}\left[|f\left(X_1(t),\cdots,X_N(t)\right)|\mathbf{1}\left(X(t)\notin[-R,R]^N\right)\right]\\ \le \|f\|_\infty \mathbb{P}_{(x_1,\cdots,x_N)}\left(X(t)\in[-R,R]^N\right)+\epsilon\mathbb{P}_{(x_1,\cdots,x_N)}\left(X(t)\notin[-R,R]^N\right),\end{aligned}$$

and also for $(x_1,\cdots,x_N),(y_1,\cdots,y_N) \in W^N$ with $P^{s,N}_{HP}(t)\left((x_1,\cdots,x_N),dz\right)$ being the law of $(X_1(t),\cdots,X_N(t))$ if $(X_1(0),\cdots,X_N(0))=(x_1,\cdots,x_N)$,

$$\begin{aligned}&|\mathbb{E}_{(x_1,\cdots,x_N)}\left[f\left(X_1(t),\cdots,X_N(t)\right)\right]-\mathbb{E}_{(y_1,\cdots,y_N)}\left[f\left(X_1(t),\cdots,X_N(t)\right)\right]|\\ &\le \|f\|_\infty \int_{W^N\cap[-R,R]^N}\left|P^{s,N}_{HP}(t)\left((x_1,\cdots,x_N),dz\right)-P^{s,N}_{HP}(t)\left((y_1,\cdots,y_N),dz\right)\right|+2\epsilon.\end{aligned}$$

Both assertions (vanishing at infinity and continuity) will follow immediately by the use of the dominated convergence theorem and the estimates on the transition density and its derivatives in the backwards variables $\partial_x^{(i)} p_t^{(N),s}(x,y)$ (for $i \ge 0$) to be presented shortly.

To be more concrete and in order to ease notation, let us first consider the most singular case $x_1 = \cdots = x_N = x$, the arguments for the others are analogous and will be explained at the end of this proof. First, note that the law of $(X_1(t),\cdots,X_N(t))$ started from $(x,\cdots,x)$ is governed by (this being well-posed is justified by the estimates presented below),

$$\lim_{x_1,\cdots,x_N\to x\mathbf{1}} e^{-\lambda_N t}\frac{\det\left(y_i^{j-1}\right)_{i,j=1}^N}{\det\left(x_i^{j-1}\right)_{i,j=1}^N}\det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N dy = e^{-\lambda_N t}\Delta_N(y)\det\left(\partial_x^{(i-1)}p_t^{(N),s}(x,y_j)\right)_{i,j=1}^N dy.$$

Hence, by expanding the determinant, we have the bound,

$$e^{-\lambda_N t}\int_{W^N\cap[-R,R]^N}\Delta_N(y)\det\left(\partial_x^{(i-1)}p_t^{(N),s}(x,y_j)\right)_{i,j=1}^N dy \lesssim C(N,t,R)\prod_{i=1}^N\int_{-R}^R|\partial_x^{(i-1)}p_t^{(N),s}(x,z)|dz.$$

From now on, to ease notation further, we write $p_t(x,y)$ for the transition density with respect to Lebesgue measure of the SDE in $\mathbb{R}$,

$$dX(t) = \sqrt{2(X^2(t)+1)}dW(t) + (\beta X(t)+\gamma)dt,$$

where β and γ are arbitrary (real) constants. We make the following smooth change of variables (in order to obtain bounded coefficients),

$$Y(t) = arsinh(X(t)) = \log\left(X(t)+\sqrt{1+X^2(t)}\right).$$

Hence, with $y = f(x) = arsinh(x)$ we have $f'(x) = \frac{1}{\sqrt{1+x^2}}$ and $f''(x) = -\frac{x}{(1+x^2)^{\frac{3}{2}}}$ and by applying

Ito's formula we obtain,

$$dY(t) = \sqrt{2}dW(t) + \left[(\beta-1)tanh(Y(t)) + \gamma sech(Y(t))\right]dt.$$

or equivalently $Y(t)$ is a diffusion in $\mathbb{R}$ with generator A given by,

$$A = \frac{d^2}{dx^2} + \left[(\beta-1)tanh(x) + \gamma sech(x)\right]\frac{d}{dx}.$$

Now, note that the coefficients are smooth with all their derivatives bounded and (obviously) the diffusion coefficient is uniformly elliptic. Thus, if we let $q_t(z,w)$ denote the transition density of $(Y(t); t \geq 0)$, from Theorem 3.3.11 of [147], we have for $i \geq 0$ with some constant C_i (depending on the ellipticity constant and the derivatives of the coefficients) the following bound,

$$|\partial_z^{(i)} q_t(z,w)| \leq \frac{C_i}{1 \wedge t^{\frac{i+1}{2}}} \exp\left(-\left(C_i t - \frac{(z-w)^2}{C_i t}\right)^-\right).$$

By applying the change of variables, the original kernel $p_t(x,y)$ for $X(t)$ is given by,

$$p_t(x,y) = q_t(f(x), f(y))\partial_y f(y) \quad \text{where} \quad f(x) = arsinh(x).$$

Now, making use of Faa-Di Bruno's formula, we obtain,

$$\partial_x^{(i)} p_t(x,y) = \sum \frac{i!}{k_1! \cdots k_i!} \partial_{f(x)}^{(k)} q_t(f(x), f(y)) \prod_{j=1}^{i} \left(\frac{\partial_x^{(j)} f(x)}{j!}\right)^{k_j} \partial_y f(y),$$

where $k = k_1 + \cdots + k_i$ and the sum is over $k_1, \cdots, k_i$ such that $k_1 + 2k_2 + \cdots + ik_i = i$.

This is a finite sum and applying the triangle inequality, we will arrive at some sufficient bound but we can in fact get the leading order terms for each of the exponentials. Observe that for $j \geq 1$,

$$|\partial_x^{(j)} f(x)| \leq \frac{c_j}{(1+x^2)^{\frac{j}{2}}} + o\left(\frac{1}{(1+x^2)^{\frac{j}{2}}}\right).$$

Hence, making use of the fact $k_1 + 2k_2 + \cdots + ik_i = i$ we get,

$$|\partial_x^{(i)} p_t(x,y)| \lesssim \left(\frac{1}{\sqrt{1+x^2}}\right)^i \left(\frac{1}{\sqrt{1+y^2}}\right) \sum_{j=0}^{i} c(j,i,t) \exp\left(-\left(C_j t - \frac{(arsinh(x) - arsinh(y))^2}{C_j t}\right)^-\right) + \text{l.o.t},$$

where l.o.t stands for lower order terms. By the continuity of $x \mapsto \partial_x^{(i)} p_t(x,y)$, the estimate above and the dominated convergence theorem the Feller property follows.

We will now treat the more general case when some of the points $(x_1^{(n)}, \cdots, x_N^{(n)})$, not necessarily all, can come together as they go to ∞ with $n \to \infty$. First, we write:

$$P_{HP}^{s,N}(t)(x_1,\cdots,x_N;y_1,\cdots,y_N) = e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)}F_t(x_1,\cdots,x_N;y_1,\cdots,y_N),$$

where,

$$F_t(x_1,\cdots,x_N;y_1,\cdots,y_N) = \det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N.$$

We can then split $(x_1^{(n)},\cdots,x_N^{(n)})$ into m blocks $(x_{i_1+\cdots+i_{j-1}+1}^{(n)},\cdots,x_{i_1+\cdots+i_j}^{(n)})$, with $i_1+\cdots+i_m = N$ and $i_0 = 0$ such that $|x_{i_1+\cdots+i_j}^{(n)} - x_{i_1+\cdots+i_j+1}^{(n)}| \geq \text{Const}$ for $j = 1,\cdots,m$ uniformly in n.

From now on we will suppress the dependence of F on $t, y_1,\cdots,y_N$ and write $F(x_1,\cdots,x_N)$. Note that, $(x_1^{(n)},\cdots,x_N^{(n)}) \to \infty$ if and only if at least one of $x_N^{(n)} \to \infty$ or $x_1^{(n)} \to -\infty$ happens and without loss of generality we assume that $x_1^{(n)} \to -\infty$. The problematic singular terms coming from the Vandermonde determinant $\Delta_N(x)$ are of course:

$$\frac{1}{\prod_{i_1+\cdots+i_{j-1}+1\leq l_1<l_2\leq i_1+\cdots+i_j}(x_{l_2}^{(n)} - x_{l_1}^{(n)})}$$

which blow up as $n \to \infty$. The crux is that these singularities are cancelled out by vanishing terms coming from $F_t(x_1,\cdots,x_N;y_1,\cdots,y_N)$.

We begin by applying the Mean Value Theorem (MVT) to the first block and we will suppress dependence on n from now on. To ease notation write $F(x_1,\cdots,x_N) = \tilde{F}(x_1,\cdots,x_k)$ where $k = i_1$ and we write ∂_l for the derivative with respect to the l^{th} variable. Then, since $\tilde{F}(x_1,\cdots,x_{k-1},x_{k-1}) = 0$, we have for some ξ_k^1 such that $x_{k-1} < \xi_k^1 < x_k$:

$$\tilde{F}(x_1,\cdots,x_{k-1},x_k) = (x_k - x_{k-1})\partial_k\tilde{F}\left(x_1,\cdots,x_{k-1},\xi_k^1\right).$$

Now write $\xi_i^0 = x_i$. Applying the MVT $(k-2)$ more times we obtain that for some $(\xi_2^1,\cdots,\xi_k^1)$ satisfying $\xi_1^0 < \xi_2^1 < \xi_2^0 < \cdots < \xi_k^1 < \xi_k^0$:

$$\tilde{F}(x_1,\cdots,x_{k-1},x_k) = \prod_{i=1}^{k-1}\left(\xi_{i+1}^0 - \xi_i^0\right)\partial_2\cdots\partial_k\tilde{F}(\xi_1^0,\xi_2^1,\cdots,\xi_{k-1}^1,\xi_k^1).$$

Iterating this procedure we finally get:

$$\tilde{F}(x_1,\cdots,x_{k-1},x_k) = \prod_{l=0}^{k-2}\prod_{i=l+1}^{k-1}(\xi_{i+1}^l - \xi_i^l)\partial_2\partial_3^2\cdots\partial_{k-1}^{k-2}\partial_k^{k-1}\tilde{F}(\xi_1^0,\xi_2^1,\cdots,\xi_{k-1}^{k-2},\xi_k^{k-1}),$$

for some ξ_i^l, $l = 0,\cdots,k-2$, $i = l+1,\cdots,k$ such that:

$$\xi_{l+1}^l < \xi_{l+2}^{l+1} < \xi_{l+2}^l < \cdots < \xi_k^{l+1} < \xi_k^l.$$

By the interlacing constraints above we observe that, for all $l = 0,\cdots,k-2$ and $i = l+$

$1,\cdots,k-1$:

$$\xi_{i+1}^{l}-\xi_{i}^{l}\leq x_{i+1}-x_{i-l}=\xi_{i+1}^{0}-\xi_{i-l}^{0}.$$

Thus, the following ratio is bounded:

$$\frac{\prod_{l=0}^{k-2}\prod_{i=l+1}^{k-1}(\xi_{i+1}^{l}-\xi_{i}^{l})}{\prod_{1\leq i<j\leq k}(x_j-x_i)}\leq 1.$$

In particular it will be uniformly bounded in n when the x's depend on n. Now we can obviously apply the argument above to each single block $(x^{(n)}_{i_1+\cdots+i_{j-1}+1},\cdots,x^{(n)}_{i_1+\cdots+i_j})$ for $j=1,\cdots,m$. Then the result follows by the uniform bounds on the transition kernel and its derivatives $\partial_x^{(j)}p_t(x,y)$; in particular we need bounds for the first $\sup_{j=1,\cdots,m} i_j-1$ derivatives. $\square$

We now arrive at the following proposition, which makes explicit the relation between the Hua-Pickrell measures and the Hua-Pickrell diffusions.

Proposition 3.14. *Let $\Re(s)>-\frac{1}{2}$. Then the probability measure $\mu_{HP}^{s,N}$ is the unique invariant measure of $P_{HP}^{s,N}(t)$.*

Proof. By making use of the reversibility of $p_t^{(N),s}(x,y)$ with respect to $m_s^{(N)}(x)$ and the fact that Δ_N is an eigenfunction of the sub-Markov Karlin-McGregor semigroup with eigenvalue $e^{\lambda_{N,s}t}$, we can obtain the invariance of $\mu_{HP}^{s,N}$ by $P_{HP}^{s,N}(t)$ as follows (here *const* denotes the same normalization constant in all equalities),

$$\int_{\mathring{W}^N} e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)}\det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N\times const\times\Delta_N^2(x)\prod_{j=1}^N(1+x_j^2)^{-\Re(s)-N}e^{2\Im(s)Arg(1+ix_j)}dx=$$

$$=const\times\Delta_N(y)\prod_{j=1}^N(1+y_j^2)^{-\Re(s)-N}e^{2\Im(s)Arg(1+iy_j)}e^{-\lambda_{N,s}t}\int_{W^N}\det\left(p_t^{(N),s}(y_i,x_j)\right)_{i,j=1}^N\Delta_N(x)dx$$

$$=const\times\Delta_N(y)\prod_{j=1}^N(1+y_j^2)^{-\Re(s)-N}e^{2\Im(s)Arg(1+iy_j)}e^{-\lambda_{N,s}t}e^{\lambda_{N,s}t}\Delta_N(y).$$

Now, by the regularity of the transition kernel $e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)}\det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N$ we show that actually $\mu_{HP}^{s,N}$ is the *unique* invariant probability measure of $P_{HP}^{s,N}(t)$. Namely, suppose we had at least two different invariant probability measures, then we would have at least two distinct *ergodic* ones which have to be mutually singular (see Lemma 2.10 and Theorem 2.11 of [65]). Now, since $\tau=\inf\{t>0:\exists\ 1\leq i<j\leq N \text{ such that } X_i(t)=X_j(t)\}=\infty$ almost surely (the system of *SDEs* (3.10) has no collisions or equivalently never hits a diagonal) then any invariant measure μ of $P_{HP}^{s,N}(t)$ does not charge ∂W^N. Hence, if μ_1,μ_2 are two (distinct)

ergodic measures there exists some Borel set A_1 so that $A_1 \not\subset \partial W^N$ and,

$$\mu_1(A_1) = 1 \text{ and } \mu_2(A_1) = 0. \tag{3.11}$$

Moreover, note that A_1 must have positive Lebesgue measure denoted $Leb(A_1) > 0$ for otherwise by the invariance of μ_1 we would have (since $P^{s,N}_{HP}(t)$ has a density $P^{s,N}_{HP}(t)(x,y)$ with respect to Lebesgue),

$$\mu_1(A_1) = \int_{W^N} \mu_1(dx) \int_{A_1} P^{s,N}_{HP}(t)(x,y)dy = 0.$$

But on the other hand, since we have the following fundamental *strict total positivity* fact,

$$e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)} \det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N > 0\ ,\ \forall\,(t,x,y) \in (0,\infty) \times \mathring{W}^N \times \mathring{W}^N,$$

which is exactly (a particular case of) the statement of Theorem 4 of [94] or see also Problem 6 and its solution on pages 158-159 of [83], we obtain that for any Borel set $\mathcal{A}$ such that $\mathcal{A} \not\subset \partial W^N$ and $Leb(\mathcal{A}) > 0$,

$$f^{(N)}_{\mathcal{A},t}(x) = \int_{\mathcal{A}} e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)} \det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N dy > 0\ ,\ \forall x \in \mathring{W}^N.$$

Thus, by the invariance of μ_i for $i = 1,2$ and the fact that they do not charge ∂W^N, we get,

$$\mu_i(\mathcal{A}) = \int_{W^N} \mu_i(dx) \int_{\mathcal{A}} e^{-\lambda_{N,s}t}\frac{\Delta_N(y)}{\Delta_N(x)} \det\left(p_t^{(N),s}(x_i,y_j)\right)_{i,j=1}^N dy = \int_{W^N} \mu_i(dx) f^{(N)}_{\mathcal{A},t}(x) > 0,$$

which contradicts (3.11) and thus we obtain uniqueness. □

3.5 Intertwinings and Boundary Feller process

In this section, we prove the main result of this chapter, proven as Corollary 3.18 below. In order to proceed, we first need to recall one of the main results of Chapter 1 that we require here. As we will see in the proof of Theorem 3.16 below, the additional contribution of this chapter, other than the quite non-trivial technical work of proving that all Markov kernels and semigroups are Feller; is a rather simple observation regarding one dimensional diffusion generators, which is actually what made it clear to the author that the method of intertwiners could be applied in this setting.

We begin by defining the Siegmund dual Hua-Pickrell diffusion (cf. Subsection

1.2.1 of Chapter 1) in $\mathbb{R}$, with infinitesimal generator denoted by $\widehat{L_s^{(n)}}$ given by,

$$\begin{aligned}\widehat{L_s^{(n)}} &= (x^2+1)\frac{d^2}{dx^2} + \left[2x - (2-2n-2\Re(s))\,x - 2\Im(s)\right]\frac{d}{dx}\\ &= (x^2+1)\frac{d^2}{dx^2} + \left[(2n+2\Re(s))\,x - 2\Im(s)\right]\frac{d}{dx},\end{aligned}$$

and where, both $-\infty$ and $+\infty$ are natural boundary points. The corresponding (non-exploding) *SDE* is given by,

$$dX(t) = \sqrt{2(X^2(t)+1)}dW(t) + \left[(2n+2\Re(s))\,X(t) - 2\Im(s)\right]dt$$

and the speed measure $\hat{m}_s^{(n)}$ with density with respect to Lebesgue measure given by,

$$\hat{m}_s^{(n)}(x) = (1+x^2)^{\Re(s)+n-1}e^{-2\Im(s)Arg(1+ix)}.$$

Propositions 1.11 and 1.12, more precisely the discussion and display (1.25) following Proposition 1.12, of Chapter 1 give the intertwining relation $\forall t > 0, N \geq 1$,

$$\mathcal{P}_s^{(N+1)}(t)\Lambda_{N,N+1} = \Lambda_{N,N+1}\hat{\mathcal{P}}_s^{(N+1)}(t), \tag{3.12}$$

where $\mathcal{P}_s^{(N+1)}(t)$ is the sub-Markov Karlin-McGregor semigroup associated to $N+1$ $L_s^{(N+1)}$-diffusions killed when they intersect or equivalently the semigroup with kernel in W^{N+1} given by,

$$\det\left(p_t^{(N+1),s}(x_i,y_j)\right)_{i,j=1}^{N+1}dy.$$

Similarly, $\hat{\mathcal{P}}_s^{(N+1)}(t)$ is the sub-Markov Karlin-McGregor semigroup associated to N $\widehat{L_s^{(N+1)}}$-diffusions having kernel (where we denote by $\hat{p}_t^{(N+1),s}$ the transition kernel of a single $\widehat{L_s^{(N+1)}}$-diffusion process),

$$\det\left(\hat{p}_t^{(N+1),s}(x_i,y_j)\right)_{i,j=1}^{N}dy,$$

and $\Lambda_{N,N+1}$ is the, not yet normalized, in particular sub-Markov, kernel,

$$\Lambda_{N,N+1}(x,dy) = \prod_{i=1}^{N}\hat{m}_s^{(N+1)}(y_i)1(y\in W^{N,N+1}(x))dy.$$

Remark 3.15. *Let us say a word on the proof of (3.12). It relies on the following relation between the transition densities of one dimensional diffusions in Siegmund duality:*

$$p_t(x,y) = \partial_y\int_x^r \hat{p}_t(y,dz).$$

One can then prove (3.12) by direct calculation. However as shown in Chapter 1, relation (3.12) follows immediately from the very structure of a certain block matrix determinant transition kernel; the reader is referred to Chapter 1 for more details and motivation behind the introduction of this block determinant kernel.

We are now ready to state and prove the key Theorem behind the construction:

Theorem 3.16. *Let $N \geq 1$ and $f \in C_0\left(W^N\right)$ then $\forall t \geq 0$,*

$$P_{HP}^{s,N+1}(t)\Lambda_N^{N+1} f = \Lambda_N^{N+1} P_{HP}^{s,N}(t) f. \tag{3.13}$$

Proof. The proof hinges on the following simple observation regarding one dimensional diffusion operators: namely an easy calculation gives that the function $\left(\hat{m}_s^{(N+1)}\right)^{-1}(x)$ is a positive eigenfunction of $\widehat{L_s^{(N+1)}}$ with eigenvalue $c_{N,s} = -2N - \Re(s)$ and the h-transform of $\widehat{L_s^{(N+1)}}$ by this eigenfunction is the $L_s^{(N)}$-diffusion. Thus, performing an h-transform of the right hand side of (3.12) by $\prod_{i=1}^N \left(\hat{m}_s^{(N+1)}\right)^{-1}(y_i)\Delta_N(y)$ which corresponds to transforming the N $\widehat{L_s^{(N+1)}}$-diffusions into N $L_s^{(N)}$-diffusions and conditioning those by the Vandermonde determinant $\Delta_N(y)$, we obtain the following equality of Markov kernels for $t > 0$ and $x \in \mathring{W}^N$,

$$(P_{HP}^{s,N+1}(t)\Lambda_N^{N+1})(x,dy) = (\Lambda_N^{N+1} P_{HP}^{s,N}(t))(x,dy).$$

Now, by using the Feller property of the kernels involved (Lemma 3.3 and Lemma 3.13), we can extend this to $t \geq 0$ and $x \in W^N$ and obtain the statement of the Theorem. ◻

Making use of Proposition 3.14, we immediately get the following corollary,

Corollary 3.17. *Let $\Re(s) > -\frac{1}{2}$ then the Hua-Pickrell measures are consistent with the links,*

$$\mu_{HP}^{s,N+1}\Lambda_N^{N+1} = \mu_{HP}^{s,N}.$$

Finally, using Theorem 3.16 above and Theorem 3.8 we readily get,

Corollary 3.18. *There exists a unique Feller semigroup $P_{HP}^{s,\infty}(t)$ on Ω that is consistent with the semigroups $\{P_N(t)\}_{N\geq 1}$, so that for $f \in C_0\left(W^N\right)$,*

$$P_{HP}^{s,\infty}(t)\Lambda_N^{\infty} f = \Lambda_N^{\infty} P_{HP}^{s,N}(t) f \text{ , } \forall t \geq 0, \ \forall N \geq 1.$$

Moreover, if $\Re(s) > -\frac{1}{2}$ the measure μ_{HP}^s is its unique invariant measure.

3.5.1 Approximation of processes on the boundary

For any Feller process encountered below, taking values in a locally compact metrizable separable space $\mathcal{X}$, we assume that we are always dealing with its cadlag modification in the space $\mathsf{D}(\mathbb{R}_+, \mathcal{X})$, of right continuous functions with left limits. In order to describe the approximation procedure, we begin by recalling some of the setup. Suppose $\{M_N\}_{N\ge1}$ is a sequence of coherent probability measures on $\{W^N\}_{N\ge1}$,

$$M_{N+1}\Lambda_N^{N+1} = M_N \ , \forall N \ge 1,$$

and let M denote the corresponding measure on Ω. We can embed W^N into Ω as follows, by defining for $x^{(N)} \in W^N$,

$$\begin{aligned}
\alpha_i^+\left(x^{(N)}\right) &= \begin{cases} \frac{\max\{x^{(N)}_{N+1-i},0\}}{N} & i = 1, \cdots, N \\ 0 & i = N+1, N+2, \cdots \end{cases}, \\
\alpha_i^-\left(x^{(N)}\right) &= \begin{cases} \frac{\max\{-x^{(N)}_{i},0\}}{N} & i = 1, \cdots, N \\ 0 & i = N+1, N+2, \cdots \end{cases}, \\
\gamma_1\left(x^{(N)}\right) &= \sum_{i=1}^{\infty} \alpha_i^+\left(x^{(N)}\right) - \sum_{i=1}^{\infty} \alpha_i^-\left(x^{(N)}\right) = \frac{x_1^{(N)} + \cdots + x_N^{(N)}}{N}, \\
\delta\left(x^{(N)}\right) &= \sum_{i=1}^{\infty} \left(\alpha_i^+\left(x^{(N)}\right)\right)^2 + \sum_{i=1}^{\infty} \left(\alpha_i^-\left(x^{(N)}\right)\right)^2 = \frac{\left(x_1^{(N)}\right)^2 + \cdots + \left(x_N^{(N)}\right)^2}{N^2}.
\end{aligned}$$

We will denote these embeddings by $\mathfrak{r}_N : W^N \hookrightarrow \Omega$ and hence we can view each M_N as a probability measure on Ω under the pushforward $(\mathfrak{r}_N)_* M_N$. Then, from sections 4 and 5 of [25], see also Section 2.1 of [130], M is the measure on Ω corresponding to the *coherent family* $\{M_N\}_{N\ge1}$ if and only if the following convergences in distribution hold as $N \to \infty$,

$$\begin{aligned}
\alpha_i^+\left(x^{(N)}\right) &\xrightarrow{d} \alpha_i^+(\omega) \ , \forall i \ge 1, \\
\alpha_i^-\left(x^{(N)}\right) &\xrightarrow{d} \alpha_i^-(\omega) \ , \forall i \ge 1, \\
\gamma_1\left(x^{(N)}\right) &\xrightarrow{d} \gamma_1(\omega), \\
\delta\left(x^{(N)}\right) &\xrightarrow{d} \delta(\omega),
\end{aligned}$$

where $x^{(N)}$ is sampled according to M_N and ω according to M. And in such a case, as before, we write,

$$\gamma_2(\omega) = \delta(\omega) - \sum_{i=1}^{\infty} \left(\alpha_i^+(\omega)\right)^2 - \sum_{i=1}^{\infty} \left(\alpha_i^-(\omega)\right)^2.$$

Now, consider a family of Feller semigroups $\{P_N(t); t \ge 0\}_{N\ge1}$ consistent with the links Λ_N^{N+1} and let $\left(X^{(N)}(t); t \ge 0\right)$ denote a realization of the corresponding Markov processes. Moreover, let $P_\infty(t)$ be the semigroup on Ω obtained by the method of the intertwiners and denote a realization of this by $(\mathsf{X}_\infty(t); t \ge 0)$. Note that, we can of course, embed spaces of paths taking values in W^N into Ω valued paths, in the obvious way and by abusing notation we write $\alpha_i^+\left(X^{(N)};t\right), \alpha_i^-\left(X^{(N)};t\right), \gamma_1\left(X^{(N)};t\right), \delta\left(X^{(N)};t\right)$ for this. Moreover, we still denote these embeddings by $\mathfrak{r}_N$.

Remark 3.19. *Note that, for any given fixed N, the process,*

$$\left(\bar{X}^{(N)}(t); t \geq 0\right) \stackrel{def}{=} \left(\mathfrak{r}_N\left(X^{(N)}\right)(t); t \geq 0\right) = \left(\alpha_i^{\pm}\left(X^{(N)};t\right), \gamma_1\left(X^{(N)};t\right), \delta\left(X^{(N)};t\right), i \in \mathbb{N}; t \geq 0\right),$$

is Markovian (as an injective function of a Markov process see for example Exercise 1.17 in Chapter 3 of [134]) and it's also clear that it is Feller. Moreover, we can easily explicitly reconstruct $\left(X^{(N)}(t); t \geq 0\right)$ from $\left(\bar{X}^{(N)}(t); t \geq 0\right)$. Towards this end we define the indexing process for $t \geq 0$,

$$\mathfrak{n}(t) = \underset{1 \leq i \leq N}{argmin}\{\alpha_i^+\left(X^{(N)};t\right) = 0\},$$

and observe that $\mathfrak{n}(t)$ is measurable with respect to the filtration generated by $\left(\alpha_i^+\left(X^{(N)};\cdot\right), 1 \leq i \leq N\right)$ up to time t. With this definition in place the following is transparent,

$$\begin{aligned}\left(X_1^{(N)}(t), \cdots, X_N^{(N)}(t); t \geq 0\right) = N\Big(& -\alpha_1^-\left(X^{(N)};t\right), \cdots, -\alpha_{N+1-\mathfrak{n}(t)}^-\left(X^{(N)};t\right), \\ & \alpha_{\mathfrak{n}(t)-1}^+\left(X^{(N)};t\right), \cdots, \alpha_1^+\left(X^{(N)};t\right); t \geq 0\Big).\end{aligned}$$

The key observation now is, that if $\{\mu_N\}_{N\geq 1}$ is a consistent family of measures then for any fixed $t \geq 0$, $\{\mu_N P_N(t)\}_{N \geq 1}$, i.e. the laws of $X^{(N)}(t)$ if $X^{(N)}(0) \stackrel{d}{=} \mu_N$, form a coherent sequence as well. This can be seen as follows,

$$\mu_{N+1} P_{N+1}(t) \Lambda_N^{N+1} = \mu_{N+1} \Lambda_N^{N+1} P_N(t) = \mu_N P_N(t),$$

and moreover, if μ is the probability measure on Ω corresponding to $\{\mu_N\}_{N\geq 1}$ then we have,

$$\mu P_\infty(t) \Lambda_N^\infty = \mu \Lambda_N^\infty P_N(t) = \mu_N P_N(t).$$

Thus, if the initial conditions converge as $N \to \infty$,

$$\begin{aligned} \alpha_i^+\left(x^{(N)};0\right) &\xrightarrow{d} \alpha_i^+(0), \forall i \geq 1, \\ \alpha_i^-\left(x^{(N)};0\right) &\xrightarrow{d} \alpha_i^-(0), \forall i \geq 1, \\ \gamma_1\left(x^{(N)};0\right) &\xrightarrow{d} \gamma_1(0), \\ \delta\left(x^{(N)};0\right) &\xrightarrow{d} \delta(0), \end{aligned}$$

where each $x^{(N)}$ is sampled according to the *coherent measures* μ_N and $\alpha_i^+(0), \alpha_i^-(0), \gamma_1(0), \delta(0)$ according to μ (we are abusing notation here, the parameter 0 really corresponds to time and has nothing to do with ω) then, for any *fixed* $t \geq 0$ we have as $N \to \infty$,

$$\begin{aligned}
\alpha_i^+\left(x^{(N)};t\right) &\xrightarrow{d} \alpha_i^+(t) \ , \forall i \geq 1, \\
\alpha_i^-\left(x^{(N)};t\right) &\xrightarrow{d} \alpha_i^-(t) \ , \forall i \geq 1, \\
\gamma_1\left(x^{(N)};t\right) &\xrightarrow{d} \gamma_1(t), \\
\delta\left(x^{(N)};t\right) &\xrightarrow{d} \delta(t),
\end{aligned}$$

where, the $\left(\alpha_i^{\pm}(t), \gamma_1(t), \delta(t)\right)$ have the law of $\mu P_\infty(t)$ (or equivalently they are just $\mathsf{X}_\infty(t)$ written out in coordinates if $\mathsf{X}_\infty(0) \stackrel{d}{=} \mu$). We state this as a proposition.

Proposition 3.20. *For each $N \geq 1$, let $\left(X^{(N)}(t); t \geq 0\right)$ be Feller processes in W^N that are consistent with the links Λ_N^{N+1} $\forall N \geq 1$. Denote by $(\mathsf{X}_\infty(t); t \geq 0)$ the Feller-Markov process on Ω obtained by the method of the intertwiners and also let as before $\left(\bar{X}^{(N)}(t); t \geq 0\right) = \left(\mathrm{r}_N\left(X^{(N)}\right)(t); t \geq 0\right)$. Finally, assume that $\{\mu_N\}_{N\geq 1}$ is a consistent family of probability measures with corresponding measure μ on Ω. Then, if $\forall N \geq 1$ $\bar{X}^{(N)}(0) \stackrel{d}{=} (\mathrm{r}_N)_* \mu_N$ and $\mathsf{X}_\infty(0) \stackrel{d}{=} \mu$ we have for any fixed $t \geq 0$,*

$$\bar{X}^{(N)}(t) \xrightarrow{d} \mathsf{X}_\infty(t) \ \text{ as } \ N \to \infty,$$

or equivalently,

$$\mathsf{X}_\infty(t) \stackrel{d}{=} w\text{–}\lim_{N\to\infty} (\mathrm{r}_N)_* \left(\mu_N P_N(t)\right),$$

where $w\text{–}\lim$ denotes the weak limit of measures.

The result above, although general might seem rather weak as a convergence statement but note however that since any point $\omega \in \Omega$ is given (by definition) by an extremal sequence of coherent probability measures Proposition 3.20 completely characterizes the abstract semigroup $P_\infty(t)$ and thus also $(\mathsf{X}_\infty(t); t \geq 0)$. We view the processes $\left(\bar{X}^{(N)}(t); t \geq 0\right)$ here, as more of a means to an end; of describing a general $(\mathsf{X}_\infty(t); t \geq 0)$ via a concrete approximation procedure from its finite N analogues. As we shall see in subsection 3.5.2 below, much stronger convergence results can be obtained on a case by case basis.

Remark 3.21. *It would still be interesting however to try to prove convergence of the semigroups in general, as in Section 3 of [29], which then, by virtue of Theorem 19.25 of [90] for example, gives weak convergence as processes in $\mathsf{D}(\mathbb{R}_+, \Omega)$ as long as the initial distributions converge (even if they not coherent).*

3.5.2 Dynamical systems on Ω coming from Dyson Brownian motions

As already mentioned in the introduction, Dyson Brownian motions (*DBM*) of different dimensions, given by the solution to the *SDEs*,

$$dX_i^N(t) = dW_i^N(t) + \sum_{j \neq i} \frac{1}{X_i^N(t) - X_j^N(t)} dt,$$

and with semigroups denoted by $P_{DBM}^N(t)$ are also consistent with the links Λ_N^{N+1}. We hence, again obtain a Feller-Markov process on Ω that however has no invariant probability measure. We now describe the boundary process explicitly. We first show that, $\forall T > 0$ we have,

$$\max_{1 \leq i \leq N} \sup_{0 \leq t \leq T} \left| \alpha_i^+ \left(X^{(N)}; t \right) - \alpha_i^+ \left(X^{(N)}; 0 \right) \right| \to 0 \text{ almost surely as } N \to \infty.$$

This can be seen as follows,

$$\begin{aligned} \frac{1}{N} \max_{1 \leq i \leq N} \sup_{0 \leq t \leq T} \left| \max\{X_{N+1-i}^{(N)}(t), 0\} - \max\{X_{N+1-i}^{(N)}(0), 0\} \right| &\leq \frac{1}{N} \max_{1 \leq i \leq N} \sup_{0 \leq t \leq T} \left| X_{N+1-i}^{(N)}(t) - X_{N+1-i}^{(N)}(0) \right| \\ &\leq \frac{1}{N} \max_{1 \leq i \leq N} \sup_{0 \leq t \leq T} \left| Y_{N+1-i}^{(N)}(t) \right| \\ &= \frac{1}{N} \max\{\sup_{0 \leq t \leq T} Y_N^{(N)}(t), -\inf_{0 \leq t \leq T} Y_1^{(N)}(t)\}, \end{aligned}$$

where $Y^{(N)}$ is an N particle *DBM* starting from the origin. But by Theorem 3.7 of [126] we have,

$$\frac{1}{\sqrt{N}} \sup_{0 \leq t \leq T} Y_N^{(N)}(t) \to 2\sqrt{T} \text{ almost surely as } N \to \infty,$$

and similarly for $-\frac{1}{\sqrt{N}} \inf_{0 \leq t \leq T} Y_1^{(N)}(t)$. The claim then follows and so since $T > 0$ was arbitrary we obtain for $i \in \mathbb{N}$,

$$\alpha_i^+(t) = \alpha_i^+(0), \forall t \geq 0.$$

Analogously, for $i \in \mathbb{N}$,

$$\alpha_i^-(t) = \alpha_i^-(0), \forall t \geq 0.$$

We now have the following equation for $\gamma_1 \left(X^{(N)}; \cdot \right)$,

$$d\gamma_1 \left(X^{(N)}; t \right) = \frac{1}{N} \sum_{i=1}^{N} dW_i^N(t) = \frac{1}{\sqrt{N}} d\beta^N(t),$$

where by Levy's characterization β^N is a standard Brownian motion and thus as $N \to \infty$,

$$\gamma_1(t) = \gamma_1(0) \, , \forall t \geq 0.$$

Finally, after an application of Ito's formula and some manipulations (see for example Step 2 of the proof of Theorem 1.1 in [131] for the details) we obtain,

$$d\delta\left(X^{(N)};t\right) = \frac{1}{N^2}\left[N^2 dt + 2\sqrt{N^2\delta\left(X^{(N)};t\right)}d\tilde{\beta}^N(t)\right] = dt + \frac{1}{N}2\sqrt{\delta\left(X^{(N)};t\right)}d\tilde{\beta}^N(t),$$

where $\tilde{\beta}^N$ is a standard Brownian motion. Thus, from Theorem 11.1.4 of [148] for example, we obtain,

$$\delta(t) = t + \delta(0),$$

and so,

$$\gamma_2(t) = t + \gamma_2(0).$$

Hence, the boundary Feller process corresponding to DBM increases the Gaussian component linearly in time while it does nothing to the rest.

On the other hand, we could have considered a stationary or Ornstein-Uhlenbeck version of *DBM*. These are given by the solutions to the SDEs,

$$dX_i^N(t) = dW_i^N(t) + \left[-cX_i^N(t) + \sum_{j \neq i} \frac{1}{X_i^N(t) - X_j^N(t)}\right]dt,$$

and with semigroups denoted by $P_{OU}^{c,N}(t)$ they are consistent with the links. For each N, we have that $P_{OU}^{c,N}(t)$ has the GUE_N ensemble with variance $\frac{1}{2c}$ as its unique invariant probability measure. Hence, the corresponding Markov process on Ω has as unique invariant measure a delta function concentrated at $\gamma_2(\omega) = \frac{1}{2c}$ with all the other coordinates $\gamma_1(\omega), \alpha_k^+(\omega), \alpha_k^-(\omega)$ being identically zero. Analogous considerations as for DBM, give the following differential equations for the $\alpha_i^{\pm}, \gamma_1$ and δ,

$$\frac{d}{dt}\alpha_i^{\pm}(t) = -c\alpha_i^{\pm}(t) \, , \, \frac{d}{dt}\gamma_1(t) = -c\gamma_1(t) \, , \, \frac{d}{dt}\delta(t) = (1 - 2c\delta(t)).$$

Solving them, we obtain,

$$\alpha_i^{\pm}(t) = \alpha_i^{\pm}(0)e^{-ct} \, , \gamma_1(t) = \gamma_1(0)e^{-ct} \, , \, \delta(t) = \frac{1}{2c}\left(1 - e^{-2ct}\right) + \delta(0)e^{-2ct},$$

and so,

$$\gamma_2(t) = \frac{1}{2c}\left(1 - e^{-2ct}\right) + \gamma_2(0)e^{-2ct}.$$

Hence, as already observed above, we can easily see that the delta measure with $\gamma_2 = \frac{1}{2c}$ and all other coordinates being 0 is the unique invariant measure and moreover the process converges exponentially fast to it.

Remark 3.22. *It is natural to try to apply the same scheme for the Hua-Pickrell diffusions. As expected, it can be seen at least formally that, in this case both the noise and the long range interactions will still be present in the limit $N \to \infty$ and we will be dealing with a truly infinite dimensional system of SDEs (ISDE). Making rigorous sense of this is not straightforward, however there is some hope that one might be able to treat this with the general theory currently being developed for such systems of ISDE by Osada and coworkers, see for example [124].*

3.6 Dynamics on the path space of the graph of spectra

The goal of this section is to construct a Markov process on the path space of the graph of spectra, such that the projection on level N evolves according to $P^{s,N}_{HP}(t)$. The motivation behind this study is to provide a relation between the discrete dynamics introduced by Borodin and Olshanski in [28] on the path space of the Gelfand-Tsetlin graph, that we will elaborate on later on (see also Chapter 5), and the constructions of this chapter.

Firstly, continuing with the graph analogy, if a *"vertex"* at level n of the *graph of spectra* corresponds to a point $\left(x_1^{(n)}, \cdots, x_n^{(n)}\right)$ in W^n, then a *path* with N steps is given by an interlacing array $\left(x_i^{(n)}, 1 \le i \le n \le N : x_i^{(n+1)} \le x_i^{(n)} \le x_{i+1}^{(n+1)}\right)$ or continuous Gelfand-Tsetlin pattern $\mathbb{GT}_c(N)$ with N levels.

For any $N \ge 1$, we can construct a Markov process on such *paths* or equivalently a Markovian evolution taking values in the space of continuous Gelfand-Tsetlin patterns $\mathbb{GT}_c(N)$, as follows, for $1 \le i \le n \le N$, until the stopping time $\mathfrak{T}_{\mathbb{GT}_c(N)}$ (see below) given by,

$$dX_i^{(n)}(t) = \sqrt{2((X_i^{(n)})^2(t)+1)}d\beta_i^{(n)}(t) + \left[(2-2n-2\mathfrak{R}(s))\,X_i^{(n)}(t) + 2\mathfrak{I}(s)\right]dt + \frac{1}{2}dK_i^{(n),-}(t) - \frac{1}{2}dK_i^{(n),+}(t),$$

where $K_i^{(n),-}$ and $K_i^{(n),+}$ are the semimartingale local times of $X_i^{(n)} - X_{i-1}^{(n-1)}$ and $X_i^{(n)} - X_i^{(n-1)}$ at 0 and $\beta_i^{(n)}$ for $1 \le i \le n \le N$ are independent standard Brownian motions. Note that the interaction is purely local and moreover that level n given level $n-1$ is *autonomous* consisting of n independent $L_s^{(n)}$-diffusions that are kept apart by the *random barriers* $\left(X_1^{(n-1)}, \cdots, X_{n-1}^{(n-1)}\right)$.

There is a slight technical issue here, that corresponds to the fact that two paths at level n (for some $n \le N$) might meet at the stopping time $\mathfrak{T}_{\mathbb{GT}_c(N)}$ given as,

$$\mathfrak{T}_{\mathbb{GT}_c(N)} = \inf\{t > 0 : \exists\, 1 \le i < j \le n \le N \text{ s.t } X_i^{(n)}(t) = X_j^{(n)}(t)\},$$

at which point we must stop the process. However, under some special initial conditions which we call *Gibbs* or *central* (see (3.14) below) $\mathfrak{T}_{\mathbb{GT}_c(N)}=\infty$ almost surely and in particular the process in $\mathbb{GT}_c(N)$ has infinite lifetime.

The construction above, although simple and natural looking, might seem a bit arbitrary. We will now make further use of some of the results of Chapter 1 (or one could use the alternative approach of Sun [149] along with Theorem 3.16 above) to obtain the following seemingly surprising fact, that under *Gibbs* initial conditions, the projections on the n^{th} level is Markovian, at the heart of which lies the tower of consistency/intertwining relations found in Theorem 3.16.

The argument proceeds by working with two levels at a time, applying Theorem 1.15 of Chapter 1 to obtain the following: Consider the two level process $\left(\left(X^{(n)}(t),X^{(n+1)}(t)\right);t\geq 0\right)$ with dynamics given by,

$$dX_i^{(n)}(t)=\sqrt{2((X_i^{(n)})^2(t)+1)}dW_i^{(n)}(t)+\left[(2-2n-2\mathfrak{R}(s))\,X_i^{(n)}(t)+2\mathfrak{I}(s)+\sum_{j\neq i}\frac{2((X_i^{(n)}(t))^2+1)}{X_i^{(n)}(t)-X_j^{(n)}(t)}\right]dt,$$

$$\begin{aligned}dX_i^{(n+1)}(t)=&\sqrt{2((X_i^{(n+1)})^2(t)+1)}d\beta_i^{(n+1)}(t)+\left[(2-2(n+1)-2\mathfrak{R}(s))\,X_i^{(n)}(t)+2\mathfrak{I}(s)\right]dt\\&+\frac{1}{2}dK_i^{(n+1),-}(t)-\frac{1}{2}dK_i^{(n+1),+}(t),\end{aligned}$$

where $K_i^{(n+1),-}$ and $K_i^{(n+1),+}$ are the semimartingale local times of $X_i^{(n+1)}-X_{i-1}^{(n)}$ and $X_i^{(n+1)}-X_i^{(n)}$ at 0 and $\{W_i^{(n)}\}_{i=1}^n$ and $\{\beta_i^{(n+1)}\}_{i=1}^{n+1}$ are independent standard Brownian motions. Then, if the process is started according to $\left(\mu(dx)\Lambda_n^{n+1}(x,dy)\right)$ (where $\mu(dx)$ is a law on $\mathring{W}^{n+1}$) there exist independent standard Brownian motions $\left(W_1^{(n+1)},\cdots,W_{n+1}^{(n+1)}\right)$, that are measurable with respect to the filtration generated by $\left(X^{(n+1)}(t);t\geq 0\right)$, such that $\left(X^{(n+1)}(t);t\geq 0\right)$ evolves as,

$$\begin{aligned}dX_i^{(n+1)}(t)=&\sqrt{2((X_i^{(n+1)})^2(t)+1)}dW_i^{(n+1)}(t)+\Bigg[(2-2(n+1)-2\mathfrak{R}(s))\,X_i^{(n+1)}(t)+2\mathfrak{I}(s)+\\&\sum_{j\neq i}\frac{2((X_i^{(n+1)}(t))^2+1)}{X_i^{(n+1)}(t)-X_j^{(n+1)}(t)}\Bigg]dt,\end{aligned}$$

started from $\mu(dx)$.

So, although the original dynamics of $\left(X^{(n+1)}(t);t\geq 0\right)$ in the two level joint evolution $\left(\left(X^{(n)}(t),X^{(n+1)}(t)\right);t\geq 0\right)$, were non-autonomous and coupled to $\left(\left(X^{(n)}(t)\right);t\geq 0\right)$, the single level projection on $\left(X^{(n+1)}(t);t\geq 0\right)$ is a Hua-Pickrell diffusion if $\left(\left(X^{(n)}(t),X^{(n+1)}(t)\right);t\geq 0\right)$ is started according to the special initial condition $\mu(dx)\Lambda_n^{n+1}(x,dy)$. Moreover, note that now,

$$\mathfrak{T}_{W^{n+1}}=\inf\{t>0:\exists\,1\leq i<j\leq n+1\text{ s.t }X_i^{(n+1)}(t)=X_j^{(n+1)}(t)\}=\infty\ \ a.s.,$$

something which is a-priori not clear, if we look at the process $\left(\left(X^{(n)}(t),X^{(n+1)}(t)\right);t\geq 0\right)$ started from an arbitrary initial condition.

Now, let $\nu_N(dx^{(N)})$ be a probability measure on $\mathring{W}^N$ and consider the following *central* or *Gibbs* measure on $\mathbb{GT}_c(N)$,

$$\nu_N(dx^{(N)}) Uniform^{x^{(N)}}_{\mathbb{GT}_c(N)}(dx^{(1)},\cdots,dx^{(N-1)}), \tag{3.14}$$

where,

$$Uniform^{x^{(N)}}_{\mathbb{GT}_c(N)}(dx^{(1)},\cdots,dx^{(N-1)}) = \frac{\prod_{j=1}^{N-1} j!}{\Delta_N(x^{(N)})}\mathbf{1}\left(x^{(1)} \prec x^{(2)} \prec \cdots \prec x^{(N-1)} \prec x^{(N)}\right)dx^{(1)}\cdots dx^{(N-1)},$$

is the uniform distribution on $\mathbb{GT}_c(N)$ with fixed bottom row $x^{(N)}$.

Iterating the two level construction, we obtain by induction, see Proposition 1.17 in Chapter 1 where this is detailed, that if the process in $\mathbb{GT}_c(N)$ is started according to the *Gibbs* measure (3.14), then the projection on the n^{th} level evolves as a Markov process, with semigroup $P^{s,n}_{HP}(t)$, started according to $\left(\nu_N\Lambda^N_n\right)(dx^{(n)})$ i.e it evolves as,

$$dX^{(n)}_i(t) = \sqrt{2((X^{(n)}_i)^2(t)+1)}dW^{(n)}_i(t) + \left[(2-2n-2\Re(s))\,X^{(n)}_i(t) + 2\Im(s) + \sum_{j\neq i}\frac{2((X^{(n)}_i(t))^2+1)}{X^{(n)}_i(t)-X^{(n)}_j(t)}\right]dt,$$

and in particular $\mathfrak{T}_{\mathbb{GT}_c(N)} = \infty$ almost surely.

Remark 3.23. *We could also more generally have replaced the measure $\nu_N(dx^{(N)})$ by an entrance law $\left(\nu^t_{N,s}(dx^{(N)}); t>0\right)$ for $P^{s,n}_{HP}(t)$ i.e. $\nu^{t_1}_{N,s}P^{s,n}_{HP}(t_2) = \nu^{t_1+t_2}_{N,s}$, see Corollary 1.16 of Chapter 1.*

We now move on, to explain a relation between the dynamics on the path space of the Gelfand-Tsetlin graph constructed by Borodin and Olshanski and the dynamics on the path space of the graph of spectra considered here: under a spacial scaling limit they give rise to the same process in continuous Gelfand-Tsetlin patterns. The reader should note that our discussion below is informal and we shall prove no Theorem, moreover the connection between the respective infinite dimensional processes on the boundaries remains mysterious.

We begin by explaining the dynamics of Borodin and Olshanski. First we will need to recall the bare minimum of definitions (a detailed study of discrete dynamics on branching graphs can be found in Chapter 5). A path of length N in the Gelfand-Tsetlin graph is given by a Gelfand-Tsetlin pattern or scheme defined as follows. We will denote by $W^n(\mathbb{Z}) = \{(x_1,\cdots,x_n)\in\mathbb{Z}^n : x_1 < \cdots < x_n\}$ ordered n-particle configurations and we will say that $y\in W^n(\mathbb{Z})$ and $x\in W^{n+1}(\mathbb{Z})$ interlace if $x_1 \le y_1 < x_2 \le \cdots \le y_n < x_{n+1}$ and abusing notation we write $y \prec x$. Then, the space of Gelfand-Tsetlin patterns of depth (or height) N is given by:

$$\mathbb{GT}(N) = \left\{\left(x^1,\cdots,x^N\right) : x^i \prec x^{i+1},\ \text{for } 1\le i\le N-1\right\}. \tag{3.15}$$

Borodin-Olshanski dynamics The dynamics were introduced in section 8 of [28] and go as follows: each of the n particles on level n has two independent exponential clocks depending on its position $x \in \mathbb{Z}$ for jumping to the right by one with rate $\lambda_n(x) = (x-(u+n-1))(x-(u'+n-1))$ and to the left by one with rate $\mu_n(x) = (x+v)(x+v')$. Here the parameters $u, u', v, v' \in \mathbb{C}$ satisfy certain constraints for the rates to be strictly positive and for the chain not to explode. In order for this Markov process to remain in $\mathbb{GT}(N)$ the particles interact through the so called push-block dynamics: There's a hierarchy for the particles, lower level ones can be thought of as heavier or more important. If the exponential clock for jumping to the right of the particle X_k^n rings first, it attempts to jump to the right by one unit. It first looks at the $(n-1)^{th}$ level to check whether it is blocked, namely if $X_k^{n-1} = X_k^n$. In case it is, nothing happens, otherwise it moves by one to the right, possibly triggering some pushing moves. Namely if the interlacing is no longer preserved with the particle labelled X_{k+1}^{n+1} then X_{k+1}^{n+1} also moves (instantaneously) to the right by one. This pushing is propagated to higher levels.

Convergence of dynamics on path space Intuitively the push-block dynamics are the discrete analogue of the local reflection interactions found in the *SDEs* above, since particles interact only when the interlacing is about to be broken. The rigorous justification of this goes through the so called Skorokhod problem and usually requires substantial technical efforts and we will not pursue it here.

What we will do however is describe the motion of individual particles on each level under a scaling limit. We will consider the following discrete to continuous scaling limit $x \rightsquigarrow x/M$ and we send $M \to \infty$ for the dynamics on the Gelfand-Tsetlin graph. Note that we just scale space and not time. Then, we formally obtain, modulo the convergence of the discrete push-block dynamics to *SDEs* with reflection, a process on the path space of the graph of spectra. Particles on level n move according to a diffusion process $(G(t); t \geq 0)$ with generator:

$$x^2 \frac{d^2}{dx^2} + (2 - 2n - (u + u' + v + v')) \frac{d}{dx}.$$

This is actually a geometric Brownian motion and is given explicitly, in terms of a standard Brownian motion $\beta(t)$:

$$G(t) = G(0) \exp\left(\sqrt{2}\beta(t) + (1 - 2n - (u + u' + v + v'))\, t\right).$$

We now perform the same spacial, continuous to continuous in this case, scaling limit $x \rightsquigarrow x/M$ with $M \to \infty$ to the Hua-Pickrell dynamics introduced above. Particles on level n will then follow a diffusion with generator:

$$x^2 \frac{d^2}{dx^2} + (2 - 2n - 2\Re(s)) \frac{d}{dx}.$$

The Markov process obtained then coincides with the one we get from the discrete to continuous limit with the identification $2\Re(s) = u + u' + v + v'$. In terms of *SDEs* with reflection this multilevel process, see also section 3.6 of [6], is given by:

$$dX_i^{(n)}(t) = \sqrt{2}|X_i^{(n)}(t)|d\beta_i^{(n)}(t) + \left[(2-2n-2\Re(s))\,X_i^{(n)}(t)\right]dt + \frac{1}{2}dK_i^{(n),-}(t) - \frac{1}{2}dK_i^{(n),+}(t).$$

3.7 Dynamics for multilevel CUE

The purpose of this section is to investigate how the results above transfer to the circle $\mathbb{T}$ under the Cayley transform. With $u = e^{i\theta}$ and $u = \frac{i-x}{i+x}$ and $x = i\frac{1-u}{1+u}$ then we have,

$$x = \tan\left(\frac{\theta}{2}\right) \text{ or } \theta = 2\tan^{-1}(x).$$

Note that, our processes never blow up to $\pm\infty$ then -1 is never attained for u and equivalently $\pm\pi$ for θ. We first find what an $L_s^{(N)}$-diffusion gets mapped to. Note that, the resulting process $\left(\theta_1^{(N)}(t), \cdots, \theta_N^{(N)}(t)\right)$ for $s = 0$ leaves CUE_N invariant and more generally $\mathfrak{C}_*(\mu_{HP}^{s,N})$ for $\Re(s) > -\frac{1}{2}$. Consider the function $f(x) = 2\tan^{-1}(x)$ and observe that, $f'(x) = \frac{2}{1+x^2}$ and $f''(x) = \frac{-4x}{(1+x^2)^2}$. Then, applying Ito's formula, we get with $u_j^{(N)}(t) = e^{i\theta_j^{(N)}(t)}$ so that $\theta_i^{(N)}(t) = 2\tan^{-1}\left(X_i^{(N)}(t)\right)$,

$$\begin{aligned}
d\theta_i^{(N)}(t) = {} & \frac{2}{((X_i^{(N)})^2(t)+1)}\Bigg[\sqrt{2((X_i^{(N)})^2(t)+1)}dW_i^{(N)}(t) + \Bigg[(-2(N-1)-2\Re(s))\,X_i^{(N)}(t) + 2\Im(s) \\
& + \sum_{j\neq i}\frac{2((X_i^{(N)})^2(t)+1)}{X_i^{(N)}(t)-X_j^{(N)}(t)}\Bigg]dt\Bigg] - \frac{1}{2}\frac{4X_i^{(N)}(t)}{((X_i^{(N)})^2(t)+1)^2}2((X_i^{(N)})^2(t)+1)dt \\
= {} & \frac{2\sqrt{2}}{\sqrt{((X_i^{(N)})^2(t)+1)}}dW_i^{(N)}(t) + \Bigg[2\left(-2(N-1)-2\Re(s)\right)\frac{X_i^{(N)}(t)}{((X_i^{(N)})^2(t)+1)} - \frac{4X_i^{(N)}(t)}{((X_i^{(N)})^2(t)+1)} \\
& + \frac{4\Im(s)}{((X_i^{(N)})^2(t)+1)} + \sum_{j\neq i}\frac{4}{X_i^{(N)}(t)-X_j^{(N)}(t)}\Bigg]dt \\
= {} & 2\sqrt{2}\cos\left(\frac{\theta_i^{(N)}(t)}{2}\right)dW_i^{(N)}(t) + \Bigg[(-4N-4\Re(s))\sin\left(\frac{\theta_i^{(N)}(t)}{2}\right)\cos\left(\frac{\theta_i^{(N)}(t)}{2}\right) \\
& + 4\Im(s)\cos^2\left(\frac{\theta_i^{(N)}(t)}{2}\right) + \sum_{j\neq i}\frac{4}{\tan\left(\frac{\theta_i^{(N)}(t)}{2}\right)-\tan\left(\frac{\theta_j^{(N)}(t)}{2}\right)}\Bigg]dt.
\end{aligned}$$

Thus, the process has generator acting on $C_c^2\left(W^N(-\pi,\pi)\right)$, twice continuously differentiable functions with compact support in W^N; this class of functions is sufficiently large to characterize the distribution of $\left(\theta_1^{(N)}(t), \cdots, \theta_N^{(N)}(t); t \geq 0\right)$ since neither $\pm\pi$ or ∂W^N are ever reached (as these correspond to explosions to $\pm\infty$ and collisions for the *SDEs* (3.10)), given by the differential operator,

$$\mathfrak{L}_s^{(N)} = 4\sum_{i=1}^{N}\cos^2\left(\frac{\theta_i}{2}\right)\partial_{\theta_i}^2 + \sum_{i=1}^{N}\Bigg[(-4N-4\Re(s))\sin\left(\frac{\theta_i}{2}\right)\cos\left(\frac{\theta_i}{2}\right)+4\Im(s)\cos^2\left(\frac{\theta_i}{2}\right)+ \sum_{j\neq i}\frac{4}{\tan\left(\frac{\theta_i}{2}\right)-\tan\left(\frac{\theta_j}{2}\right)}\Bigg]\partial_{\theta_i}.$$

This can be written as an h-transform, as follows,

$$\mathfrak{L}_s^{(N)} = h_N^{-1}(\theta)\circ\sum_{i=1}^{N}L_{\theta_i}^{s,(N)}\circ h_N(\theta) - const_{N,s},$$

where the one dimensional diffusion operators are given by,

$$L_{\theta_i}^{s,(N)} = 4\cos^2\left(\frac{\theta_i}{2}\right)\frac{d^2}{d\theta_i^2}+\left[(-4N-4\Re(s))\sin\left(\frac{\theta_i}{2}\right)\cos\left(\frac{\theta_i}{2}\right)+4\Im(s)\cos^2\left(\frac{\theta_i}{2}\right)\right]\frac{d}{d\theta_i},$$

and the positive eigenfunction h_N,

$$h_N(\theta) = \prod_{1\le i<j\le N}\left(\tan\left(\frac{\theta_j}{2}\right)-\tan\left(\frac{\theta_i}{2}\right)\right).$$

We proceed to check explicitly that, for $s=0$, $\mathfrak{L}_0^{(N)}$ indeed leaves CUE_N invariant. Of course, the same argument works for any $\Re(s) > -\frac{1}{2}$. First observe that, with $s=0$ the $L^{0,(N)}$-diffusion has invariant measure on $[-\pi,\pi]$ given by,

$$const\times\cos^{2N-2}\left(\frac{\theta}{2}\right)d\theta.$$

Thus, the multidimensional process with generator $\mathfrak{L}_0^{(N)}$ has invariant measure,

$$const\times\prod_{i=1}^{N}\cos^{2N-2}\left(\frac{\theta_i}{2}\right)\prod_{1\le i<j\le N}\left(\tan\left(\frac{\theta_j}{2}\right)-\tan\left(\frac{\theta_i}{2}\right)\right)^2 d\theta_1\cdots d\theta_N,$$

which is easily seen to be equal to CUE_N (given now in terms of the eigenangles),

$$const\times\prod_{1\le i<j\le N}\sin^2\left(\frac{\theta_j-\theta_i}{2}\right)d\theta_1\cdots d\theta_N.$$

3.8 Matrix Hua-Pickrell Process

In this final short section, we define a matrix process with its eigenvalues evolving according to (3.10) and leaving the matrix Hua-Pickrell measure $\mathsf{M}_{HP}^{s,N}(dX)$, with $\Re(s) > -\frac{1}{2}$, defined in (3.9) invariant. This process will play a key role in the study undertaken in Chapter 4.

So, let $\left(\boldsymbol{B}^{(k)}_t; t \geq 0\right)$, for $k = 1, 2$, be two $N \times N$ matrices with entries independent standard Brownian motions. Moreover, define $(\boldsymbol{W}_t; t \geq 0)$ by $\boldsymbol{W}_t = \boldsymbol{B}^{(1)}_t + i\boldsymbol{B}^{(2)}_t$ and let $h : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. Consider the following stochastic process $(\boldsymbol{X}_t; t \geq 0)$, taking values in the space of $N \times N$ Hermitian matrices and verifying the matrix valued *SDE*,

$$d\boldsymbol{X}_t = g(\boldsymbol{X}_t) d\boldsymbol{W}_t h(\boldsymbol{X}_t) + h(\boldsymbol{X}_t) d\boldsymbol{W}^*_t g(\boldsymbol{X}_t) + \left(b(\boldsymbol{X}_t) + \alpha Tr\left(\boldsymbol{X}_t\right) \boldsymbol{I}\right) dt, \tag{3.16}$$

where $h(\boldsymbol{X}_t), g(\boldsymbol{X}_t), b(\boldsymbol{X}_t)$ are defined spectrally. Define the function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by,

$$G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x).$$

Denote by $(\Lambda_t; t \geq 0) = (\lambda_1(t), \cdots, \lambda_N(t); t \geq 0)$, the projection on the eigenvalues of $(\boldsymbol{X}_t; t \geq 0)$. Then if $\boldsymbol{X}_0$ has distinct eigenvalues almost surely we obtain the following *closed* (note there is no dependence on the eigenvectors) system of *SDEs* for the eigenvalues where the $\{\beta_i\}_{i=1}^N$ are independent standard (real) Brownian motions,

$$d\lambda_i(t) = 2h(\lambda_i(t))g(\lambda_i(t))d\beta_i(t) + \left(b(\lambda_i(t)) + \alpha \sum_{k=1}^{N} \lambda_k(t) + 2 \sum_{k \neq i} \frac{G\left(\lambda_i(t), \lambda_k(t)\right)}{\lambda_i(t) - \lambda_k(t)}\right) dt,$$

up to the first collision time $\tau = \inf\{t \geq 0 : \exists\ i, j \text{ such that } \lambda_i(t) = \lambda_j(t)\}$. This is essentially Theorem 4 of [77], with the only variation being that, we have the extra drift term $\alpha Tr\left(\boldsymbol{X}_t\right) \boldsymbol{I}$ which obviously gives the contribution $\alpha \sum_{k=1}^{N} \lambda_k(t)$ in the drift of the *SDEs* for the eigenvalues.

We now specialize to the case of interest and we take,

$$h(x) = \sqrt{\frac{1 + x^2}{2}}, g(x) \equiv 1, b(x) = (1 - N - 2\Re(s))x + 2\Im(s), \alpha = 1,$$

so that $(\boldsymbol{X}_t; t \geq 0)$ satisfies,

$$d\boldsymbol{X}_t = d\boldsymbol{W}_t \sqrt{\frac{I + \boldsymbol{X}_t^2}{2}} + \sqrt{\frac{I + \boldsymbol{X}_t^2}{2}} d\boldsymbol{W}^*_t + \left[(-N - 2\Re(s))\boldsymbol{X}_t + 2\Im(s)\boldsymbol{I} + Tr\left(\boldsymbol{X}_t\right) \boldsymbol{I}\right] dt. \tag{3.17}$$

With some simple algebra, using the fact,

$$\sum_{k \neq i} \frac{\lambda_k^2(t)}{\lambda_i(t) - \lambda_k(t)} = \sum_{k \neq i} \frac{\lambda_i^2(t)}{\lambda_i(t) - \lambda_k(t)} - (N - 2)\lambda_i(t) - \sum_{k=1}^{N} \lambda_k(t),$$

we obtain the system of *SDEs* (3.10),

$$d\lambda_i(t) = \sqrt{2(1+\lambda_i^2(t))}d\beta_i(t) + \left(2\Im(s) + (2-2N-2\Re(s))\,\lambda_i(t) + \sum_{j\neq i}\frac{2\left(1+\lambda_i^2(t)\right)}{\lambda_i(t)-\lambda_j(t)}\right)dt.$$

Thus, the eigenvalues $(\Lambda_t; t \geq 0)$ of $(\boldsymbol{X}_t; t \geq 0)$ form a Hua-Pickrell diffusion. Moreover, since the system of *SDEs* (3.10) has no collisions and does not explode we also get $\tau = \infty$ almost surely (this again can be seen in a couple of ways in analogy to Proposition 3.13 namely either using Theorem 2.2 of [78], which amounts to a classical argument due to McKean, or the fact that the process is a Doob h-transform of identical one dimensional diffusions killed when they intersect).

Now, in order to see that (3.17) has a unique strong solution the argument is the same as in the Wishart (see Theorem 2, also Remark 4 (b) in [41]) or Jacobi (see chapter 9 of [60]) matrix diffusion cases. Namely since the function $z \mapsto \sqrt{z}$ is analytic on the set of *strictly* positive definite matrices (see Chapter 5 paragraph 22 page 134 of [137] and obviously $z = I + X^2$ is such) and moreover the drift coefficients are Lipschitz then we obtain a unique strong solution to (3.17), by Theorem 3.1 page 164 of [82] for example.

Finally, to get the invariance of $\mathsf{M}^{s,N}_{HP}(dX)$ observe that this follows from the $\mathbb{U}(N)$ invariance of $(\boldsymbol{X}_t; t \geq 0)$ and the fact that $(\Lambda_t; t \geq 0)$, by Proposition 3.14 has $\mu^{s,N}_{HP}$, with $\Re(s) > -\frac{1}{2}$, as its unique invariant measure. To see the $\mathbb{U}(N)$ invariance of $(\boldsymbol{X}_t; t \geq 0)$, define for $U \in \mathbb{U}(N)$ $(\boldsymbol{Y}_t; t \geq 0) = (U^*\boldsymbol{X}_t U; t \geq 0)$ and observe that $(\boldsymbol{Y}_t; t \geq 0)$ also satisfies (3.17),

$$d\boldsymbol{Y}_t = d\tilde{\boldsymbol{W}}_t\sqrt{\frac{I+\boldsymbol{Y}_t^2}{2}} + \sqrt{\frac{I+\boldsymbol{Y}_t^2}{2}}d\tilde{\boldsymbol{W}}_t^* + \left[(1-N-2\Re(s))\boldsymbol{Y}_t + 2\Im(s)\boldsymbol{I} + Tr\,(\boldsymbol{Y}_t)\,\boldsymbol{I}\right]dt,$$

with $\left(\tilde{\boldsymbol{W}}_t; t \geq 0\right) = (U^*\boldsymbol{W}_t U; t \geq 0) \overset{law}{=} (\boldsymbol{W}_t; t \geq 0)$ by unitary invariance of Brownian motion, from which, if moreover $U^*\boldsymbol{X}_0 U \overset{law}{=} \boldsymbol{X}_0$, the conclusion follows.

We now give an alternative and rather neat proof for the fact that the semigroup $P^{s,N}_{HP}(t)$ has the Feller property, by appealing to known results. Since $\boldsymbol{X}_t$ solves an SDE with globally Lipschitz coefficients it is well known that it has the Feller property, see for example Theorem 19.9 of [139]. We denote by $\mathcal{S}^N(t)$ its semigroup. Note that the presence of the repulsive singular term does not allow us to apply this result directly to the eigenvalues. Moreover, observe that $f \mapsto f \circ \mathsf{eval}_N$ maps $C_0\left(W^N\right)$ to $C_0\left(H(N)\right)$.

Proposition 3.24. *The semigroup $P^{s,N}_{HP}(t)$, associated to $\mathsf{eval}_N(\boldsymbol{X}_t)$, has the Feller property.*

Proof. From the fact that the eigenvalue evolution is autonomous we obtain that $\forall f : W^N \to \mathbb{R}$ we have:

$$\mathcal{S}^N(t)\left(f \circ \mathsf{eval}_N\right)(H) \text{ only depends on } H \text{ through } \mathsf{eval}_N(H).$$

Namely, $\mathsf{eval}_N(\boldsymbol{X}_t)$ only depends on H through $\mathsf{eval}_N(\boldsymbol{X}_0 = H)$. Thus, if $x = \mathsf{eval}_N(H)$ we have:

$$\left[P^{s,N}_{HP}(t)f\right](x) = \left[\mathcal{S}^N(t)f \circ \mathsf{eval}_N\right](H) = \left[\mathcal{S}^N(t)f \circ \mathsf{eval}_N\right](U^*xU)\ , \forall U \in \mathbb{U}(N),$$

where $\mathbb{U}(N)$ is the N-dimensional unitary group. We proceed to check the Feller property. Since $x_n \to x \implies U^*x_nU \to U^*xU$ we get:

$$\left[P^{s,N}_{HP}(t)f\right](x_n) = \left[\mathcal{S}^N(t)f \circ \mathsf{eval}_N\right](U^*x_nU) \to \left[\mathcal{S}^N(t)f \circ \mathsf{eval}_N\right](U^*xU) = \left[P^{s,N}_{HP}(t)f\right](x).$$

Moreover, since $x_n \to \infty \implies U^*x_nU \to \infty$ and $\left[\mathcal{S}^N(t)f \circ \mathsf{eval}_N\right] \in C_0\left(H(N)\right)$ we get:

$$\left[P^{s,N}_{HP}(t)f\right](x_n) \to 0 \ \text{ as } \ x_n \to \infty.$$

Finally we have continuity at $t = 0$:

$$\lim_{t\to 0}\left[P^{s,N}_{HP}(t)f\right](x) = \lim_{t\to 0}\left[\mathcal{S}^N(t)f \circ \mathsf{eval}_N\right](U^*xU) = \left[f \circ \mathsf{eval}_N\right](U^*xU) = f(x).$$

The proposition is fully proven. □

Before closing, we remark that under an application of the Cayley transform we obtain a process $(\boldsymbol{U}(t); t \geq 0)$ on the unitary group $\mathbb{U}(N)$ given by,

$$\boldsymbol{U}(t) = \mathfrak{C}(\boldsymbol{X})(t) = \frac{i - \boldsymbol{X}(t)}{i + \boldsymbol{X}(t)} \in \mathbb{U}(N),$$

which has eigenvalues evolving according to $\left(e^{i\theta_1^{(N)}(t)}, \cdots, e^{i\theta_N^{(N)}(t)}; t \geq 0\right)$.

Remark 3.25. *In the special case $s = 0$ note that $(\boldsymbol{U}(t); t \geq 0)$ is a $\mathbb{U}(N)$ valued process that the projection on its eigenvalues leaves CUE_N invariant but itself is not unitary Brownian motion (and thus neither its spectrum follows circular Dyson Brownian motion abbreviated cDBM). In fact given that cDBM can wrap around $\mathbb{T}$ such a multilevel construction of an interlacing process where the number of particles increases by one on each level does not seem possible (see section 4 of [107] for example where a coupling is given for n and n particles of cDBM).*

Chapter 4

Matrix Bougerol identity

4.1 Introduction

We begin this introduction, by recalling Bougerol's celebrated identity, first established in [37] in his study of convolution powers of probabilities on certain solvable groups. Let $(\beta_t; t \geq 0)$ and $(\gamma_t; t \geq 0)$ be two independent standard Brownian motions starting from 0. Then, for *fixed* $t \geq 0$, we have the following equalities in law,

$$\sinh(\beta_t) \stackrel{\text{law}}{=} \int_0^t e^{\beta_s} d\gamma_s \stackrel{\text{law}}{=} \gamma\left(\int_0^t e^{2\beta_s} ds\right). \tag{4.1}$$

Moreover, if we denote by $\left(\beta_t^{(-\nu)}; t \geq 0\right)$ and $\left(\gamma_t^{(-\mu)}; t \geq 0\right)$ two independent standard Brownian motions with drifts $-\nu$ and $-\mu$ respectively, the law of the functional, for $\nu > 0$,

$$\int_0^\infty e^{\beta_t^{(-\nu)}} d\gamma_t^{(-\mu)} \tag{4.2}$$

has density, with respect to Lebesgue measure, given by,

$$f_{\nu,\mu}(x) = \mathrm{c}_{\nu,\mu} \frac{e^{-2\mu \arctan(x)}}{(1+x^2)^{\nu+\frac{1}{2}}}.$$

Note that this belongs to the much-studied type IV family of Pearson distributions. Both these statements, have been given simple and quite elegant diffusion theoretic proofs by Marc Yor and co-authors in [4] and [13] respectively (see also Marc Yor's monograph [171] and the survey [154] for more recent developments). The purpose of this chapter is to obtain the Hermitian matrix analogues of these results. We will establish these by adapting the strategy in the references above to the matrix setting. The real crux here, is understanding what the right matrix analogue should be.

We should also mention that, Marc Yor had an ongoing program for some time, trying to obtain higher dimensional generalizations of Bougerol's identity and study their ramifications ([38]). In the last few years, some interesting progress was made in his joint work with Bertoin and Dufresne ([16]), where a generalization involving a (still) one-dimensional process and its local time was discovered. However, our contribution provides the first truly multi-dimensional extension, moreover making a connection between stochastic analysis and the celebrated Hua-Pickrell measures coming from random matrix theory and harmonic analysis on groups.

Before continuing, let us explain a bit further the initial motivation behind the study undertaken here. There is a closely related and equally well-known identity in one dimension, originally proven by Dufresne in [61]: Consider the functional,

$$a_t^{(-\nu)} = \int_0^t e^{2\beta_s^{(-\nu)}} ds.$$

Then, for $\nu > 0$,

$$a_\infty^{(-\nu)} \overset{\text{law}}{=} \frac{1}{2\xi_\nu} \tag{4.3}$$

where ξ_ν is a Gamma distributed random variable with density $\frac{1}{\Gamma(\nu)} x^{\nu-1} e^{-x}$. Recently, Rider and Valko in [135] have proven a matrix version of this result, obtaining in place of an inverse Gamma random variable, the inverse Wishart laws. The present chapter grew out of my attempt, to both better understand their result and investigate whether other well known matrix laws can be constructed by this diffusion theoretic approach, or "Dufresne procedure" as referred to in [135]. We finally note that, the second equality in law in (4.1), obtained by a time-change, that links Bougerol's and Dufresne's (one-dimensional) identities, does not appear to have a matrix counterpart.

In order to proceed to state our results, we first need to introduce the Hermitian analogues of the Pearson distribution, of $\left(e^{\beta_t}; t \ge 0\right)$ and $(\sinh(\beta_t); t \ge 0)$.

We consider the following measure that was introduced in Section 3.3 of the previous chapter, denoted by $\mathsf{M}_{HP}^{s,N}$, on the space $\mathbf{H}(N)$, of $N \times N$ Hermitian matrices, with s being a complex parameter such that $\Re(s) > -\frac{1}{2}$,

$$\mathsf{M}_{HP}^{s,N}(d\boldsymbol{X}) = \text{const} \times \det\left((\boldsymbol{I} + i\boldsymbol{X})^{-s-N}\right) \det\left((\boldsymbol{I} - i\boldsymbol{X})^{-\bar{s}-N}\right) \times d\boldsymbol{X}, \tag{4.4}$$

where $d\boldsymbol{X}$ denotes Lebesgue measure on $\mathbf{H}(N)$. The restriction $\Re(s) > -\frac{1}{2}$ is so that the measure $\mathsf{M}_{HP}^{s,N}$ can be normalized to a probability measure. Its significance in terms of the stochastic processes we shall consider will also be clarified in Lemma 4.5 below (see also Section 3.8).

As observed in Section 3.3 of the previous chapter, using the Weyl integration formula, if we look at the radial part of $\mathsf{M}_{HP}^{s,N}(d\boldsymbol{X})$ we get a probability measure on the Weyl chamber $W^N = \{(x_1, \cdots, x_N) \in \mathbb{R}^N : x_1 \le x_2 \le \cdots \le x_N\}$ of log-gas type, which we will denote by $\mu_{HP}^{s,N}$, and is given explicitly by,

$$\begin{aligned}\mu_{HP}^{s,N}(dx) &= \text{const} \times \Delta_N^2(x) \prod_{j=1}^{N} (1+ix_j)^{-s-N}(1-ix_j)^{-\bar{s}-N} dx_j \\ &= \text{const} \times \Delta_N^2(x) \prod_{j=1}^{N} (1+x_j^2)^{-\Re(s)-N} e^{2\Im(s)\arg(1+ix_j)} dx_j,\end{aligned} \tag{4.5}$$

where $x = (x_1, \cdots, x_N)$ and $\Delta_N(x) = \prod_{1\le i<j\le N}(x_j - x_i)$ is the Vandermonde determinant.

Before introducing our stochastic dynamics, we briefly recall (from the previous chapter) some of the history of the measures $\mathsf{M}_{HP}^{s,N}$. They were first introduced by Hua Luogeng in the 50's in his monograph [81] on harmonic analysis in several complex variables and were later in the 80's rediscovered independently by Pickrell [128] in the context of Grassmann manifolds. Around the turn of the millennium, they were further studied by Neretin in [108] and Borodin and Olshanski investigated their $N \to \infty$ limits as determinantal point processes in [25]. The reader is referred to [25] and the more recent study [43] for more of their truly remarkable properties.

We now move on to the matrix stochastic processes we will be dealing with. First some notation. We will denote by $\boldsymbol{A}^\dagger$ the complex conjugate of a matrix $\boldsymbol{A}$ and in case it is invertible we write $\boldsymbol{A}^{-\dagger}$ for $\left(\boldsymbol{A}^\dagger\right)^{-1}$ and also write $\mathrm{Tr}(\boldsymbol{A})$ for the trace of $\boldsymbol{A}$. Throughout this chapter, $(\boldsymbol{W}_t; t \ge 0)$ will be an $N \times N$ complex Brownian matrix. More precisely, its entries consist of independent (scalar) complex Brownian motions.

We will denote by $\left(\boldsymbol{M}_t^{(\nu)}; t \ge 0\right)$ the matrix analogue of the exponential of complex Brownian motion with drift ν (the choice of the diffusivity constant is dictated once we fix the normalization of the equation (4.6) below), given by the solution to the following matrix Stochastic Differential Equation (SDE), starting from $\boldsymbol{M}_0^{(\nu)} = \boldsymbol{I}$,

$$d\boldsymbol{M}_t^{(\nu)} = \frac{1}{\sqrt{2}} \boldsymbol{M}_t^{(\nu)} d\boldsymbol{W}_t + \nu \boldsymbol{M}_t^{(\nu)} dt.$$

Moreover, consider the following matrix SDE (that we first introduced in Section 3.8) taking values in $\mathbf{H}(N)$ (if $\boldsymbol{X}_0 \in \mathbf{H}(N)$), where $(\boldsymbol{\Gamma}_t; t \ge 0)$ denotes a complex Brownian matrix,

$$d\boldsymbol{X}_t = d\boldsymbol{\Gamma}_t \sqrt{\frac{\boldsymbol{I} + \boldsymbol{X}_t^2}{2}} + \sqrt{\frac{\boldsymbol{I} + \boldsymbol{X}_t^2}{2}} d\boldsymbol{\Gamma}_t^\dagger + \left[(-N - 2\Re(s))\boldsymbol{X}_t + 2\Im(s)\boldsymbol{I} + \mathrm{Tr}\left(\boldsymbol{X}_t\right)\boldsymbol{I}\right] dt. \tag{4.6}$$

This is a Hermitian analogue of (a general version of) $\sinh(\beta_t)$. To see the analogy more clearly, note that,

$$d\sinh(\beta_t) = \left(1 + \sinh^2(\beta_t)\right)^{\frac{1}{2}} d\beta_t + \frac{1}{2}\sinh(\beta_t) dt.$$

Hence, to arrive at (4.6) we simply replaced the scalar (quadratic, with no real roots) diffusion and (linear) drift coefficients by their (symmetrized) matrix analogues. The appearance of the trace drift term is natural and can partly be explained by the calculations required in Propositions 4.6 and 4.7 below. Moreover, our choice of both drift and diffusivity constants, is so that (4.6) has both $\mathsf{M}_{HP}^{s,N}$ as its unique invariant measure and its eigenvalue evolution satisfies a stochastic equation with a certain normalization; this is made precise in Proposition 4.7 and its proof (see also Section 3.8).

One final piece of notation; we will write throughout $\left(\boldsymbol{B}_t^{(\mu)}; t \geq 0\right)$ for a drifting complex Brownian matrix with drift $\mu \in \mathbb{R}$, given by,

$$\boldsymbol{B}_t^{(\mu)} = \boldsymbol{B}_t + \mu \boldsymbol{I} t$$

for a complex Brownian matrix $(\boldsymbol{B}_t; t \geq 0)$ which is *independent* of $(\boldsymbol{W}_t; t \geq 0)$.

We are now ready to state our two main results. First, the law of the Hermitian analogue of the functional (4.2), is given by the Hua-Pickrell measure $\mathsf{M}_{HP}^{s,N}$.

Theorem 4.1. *Let $\Re(s) > -\frac{1}{2}$. With $\nu = \Re(s) + \frac{N}{2}, \mu = \sqrt{2}\Im(s)$, then,*

$$\int_0^\infty \boldsymbol{M}_t^{(-\nu)} \left(\frac{d\boldsymbol{B}_t^{(\mu)} + d\left(\boldsymbol{B}_t^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_t^{(-\nu)}\right)^\dagger \tag{4.7}$$

is distributed as $\mathsf{M}_{HP}^{s,N}$.

Remark 4.2. *Comparing with [135], the matrix analogue of Dufresne's identity is given by,*

$$\int_0^\infty \boldsymbol{M}_t^{(-\nu)} dt \left(\boldsymbol{M}_t^{(-\nu)}\right)^\dagger$$

which is distributed as an inverse Wishart random matrix. To obtain the Hua-Pickrell measures, we have replaced the dt integration by a stochastic integral with respect to an independent (drifting) Hermitian Brownian motion, $\left(\boldsymbol{B}_t^{(\mu)} + \left(\boldsymbol{B}_t^{(\mu)}\right)^\dagger; t \geq 0\right)$.

Finally, we have the following Hermitian version of Bougerol's identity (4.1).

Theorem 4.3. *With $\nu = \Re(s) + \frac{N}{2}, \mu = \sqrt{2}\Im(s)$, denote by $\tilde{\boldsymbol{X}}_t^{\mu,\nu}$ the unique solution of (4.6) starting from the $\mathbf{0}$ matrix. Then, for fixed $t > 0$,*

$$\tilde{\boldsymbol{X}}_t^{\mu,\nu} \overset{law}{=} \int_0^t \boldsymbol{M}_u^{(-\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(-\nu)}\right)^\dagger. \tag{4.8}$$

4.2 Preliminaries, Auxiliary results and Proofs of Theorems

As in the introduction, we denote by $\left(\boldsymbol{M}_t^{(\nu)}; t \geq 0\right)$ the matrix analogue of the exponential of complex Brownian motion with drift ν (and diffusivity $\frac{1}{\sqrt{2}}$), starting from $\boldsymbol{M}_0^{(\nu)} = \boldsymbol{I}$,

$$d\boldsymbol{M}_t^{(\nu)} = \frac{1}{\sqrt{2}}\boldsymbol{M}_t^{(\nu)} d\boldsymbol{W}_t + \nu \boldsymbol{M}_t^{(\nu)} dt,$$
$$d\left(\boldsymbol{M}_t^{(\nu)}\right)^\dagger = \frac{1}{\sqrt{2}} d\boldsymbol{W}_t^\dagger \left(\boldsymbol{M}_t^{(\nu)}\right)^\dagger + \nu \left(\boldsymbol{M}_t^{(\nu)}\right)^\dagger dt.$$

A simple application of Itô's formula gives the following SDE for $\left(\det\left(\boldsymbol{M}_t^{(\nu)}\right); t \geq 0\right)$,

$$d \det\left(\boldsymbol{M}_t^{(\nu)}\right) = \det\left(\boldsymbol{M}_t^{(\nu)}\right)\left(\frac{1}{\sqrt{2}} \operatorname{tr}\left(d\boldsymbol{W}_t\right) + \nu N dt\right).$$

Solving it, we get,

$$\det\left(\boldsymbol{M}_t^{(\nu)}\right) = \exp\left(\frac{1}{\sqrt{2}} \operatorname{tr}\left(\boldsymbol{W}_t\right) + \nu N t\right).$$

Thus, $\left(\boldsymbol{M}_t^{(\nu)}; t \geq 0\right)$ is almost surely invertible. Moreover, by applying Itô's formula to the identity $\boldsymbol{M}_t^{(\nu)}\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} = \boldsymbol{I}$, we easily obtain the following description of the dynamics of its inverse $\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1}; t \geq 0\right)$,

$$d\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} = -\frac{1}{\sqrt{2}} d\boldsymbol{W}_t \left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} - \nu \left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} dt,$$
$$d\left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger} = -\frac{1}{\sqrt{2}} \left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger} d\boldsymbol{W}_t^\dagger - \nu \left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger} dt.$$

We will also need the notion and a precise description of the evolution of the *time-reversal* of $\left(\boldsymbol{M}_t^{(\nu)}; t \geq 0\right)$. For $T \geq 0$ fixed, we will denote this time-reversed process by $\left(\boldsymbol{N}_t^{(\nu)}; 0 \leq t \leq T\right) = \left(\left(\boldsymbol{M}_T^{(\nu)}\right)^{-1} \boldsymbol{M}_{T-t}^{(\nu)}; 0 \leq t \leq T\right)$. Then, we have the following lemma.

Lemma 4.4. *$\left(\boldsymbol{N}_t^{(\nu)}; 0 \leq t \leq T\right)$ satisfies,*

$$d\boldsymbol{N}_t^{(\nu)} = \frac{1}{\sqrt{2}} \boldsymbol{N}_t^{(\nu)} d\tilde{\boldsymbol{W}}_t - \nu \boldsymbol{N}_t^{(\nu)} dt,$$

for a complex Brownian matrix $\tilde{\boldsymbol{W}}$. In particular, it is distributed as $\left(\boldsymbol{M}_t^{(-\nu)}; 0 \leq t \leq T\right)$ starting from $\boldsymbol{I}$.

Furthermore, we have the following result for the rate of growth of $\left(\boldsymbol{M}_t^{(-\nu)}; t \geq 0\right)$ as $t \to \infty$; this ensures the convergence of the various matrix integrals we have encountered under the assumption $\Re(s) > -\frac{1}{2}$.

Lemma 4.5. *Let* $\left(\eta_1^{(-\nu)}(t) \leq \cdots \leq \eta_N^{(-\nu)}(t); t \geq 0\right)$ *denote the squared singular values of* $\left(\boldsymbol{M}_t^{(-\nu)}; t \geq 0\right)$. *Then, almost surely,*

$$\lim_{t \to \infty} \frac{1}{t} \log \eta_N^{(-\nu)}(t) \leq -2\nu + N - 1.$$

In particular, if $\nu = \Re(s) + \frac{N}{2}$ *for* $\Re(s) > -\frac{1}{2}$ *we have,*

$$\lim_{t \to \infty} \frac{1}{t} \log \eta_N^{(-\nu)}(t) < 0$$

and hence, for any matrix norm $\|\cdot\|$ *we have,*

$$\lim_{t \to \infty} \frac{1}{t} \log \left\|\boldsymbol{M}_t^{(-\nu)}\right\| < 0, \text{ almost surely.}$$

It is a remarkable fact, that the solution of (4.6), for any initial condition $\boldsymbol{X}_0$, can be written out explicitly:

Proposition 4.6. *With* $\nu = \Re(s) + \frac{N}{2}, \mu = \sqrt{2}\Im(s)$, *then the unique strong solution of (4.6), starting from* $\boldsymbol{X}_0 \in \boldsymbol{H}(N)$ *is given explicitly by,*

$$\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} \left[\boldsymbol{X}_0 + \int_0^t \boldsymbol{M}_u^{(\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}}\right) \left(\boldsymbol{M}_u^{(\nu)}\right)^\dagger\right] \left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger}. \tag{4.9}$$

The final ingredient that we will make use of is the following.

Proposition 4.7. *Let* $\Re(s) > -\frac{1}{2}$. *Then, the unique strong solution* $(\boldsymbol{X}_t; t \geq 0)$ *to (4.6) has* $\mathsf{M}_{HP}^{s,N}$ *as its unique invariant measure.*

We are now in position to quickly prove our two main results.

Proof of Theorem 4.3. This follows immediately from Proposition 4.6, by making the change of variables $u \mapsto t - u$, using the time-reversal Lemma 4.4 for $\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} \boldsymbol{M}_{t-u}^{(\nu)}; 0 \leq u \leq t\right)$ and finally noting invariance under time-reversal of the matrix Brownian motion $\boldsymbol{B}$. □

Proof of Theorem 4.1. From Theorem 4.3 we have that,

$$\tilde{\boldsymbol{X}}_t^{\mu,\nu} \overset{law}{=} \int_0^t \boldsymbol{M}_u^{(-\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}}\right) \left(\boldsymbol{M}_u^{(-\nu)}\right)^\dagger. \tag{4.10}$$

Moreover, by Lemma 4.5 we have that for $\Re(s) > -\frac{1}{2}$ almost surely,

$$\int_0^t \boldsymbol{M}_u^{(-\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}}\right) \left(\boldsymbol{M}_u^{(-\nu)}\right)^\dagger \underset{t \to \infty}{\longrightarrow} \int_0^\infty \boldsymbol{M}_u^{(-\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}}\right) \left(\boldsymbol{M}_u^{(-\nu)}\right)^\dagger.$$

Thus, we have the following convergence in law (in fact for any initial condition $\boldsymbol{X}_0 \in \mathbf{H}(N)$ and not just for the $\mathbf{0}$ matrix),

$$\tilde{\boldsymbol{X}}_t^{\mu,\nu} \xrightarrow[t\to\infty]{\text{law}} \int_0^\infty \boldsymbol{M}_u^{(-\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(-\nu)}\right)^\dagger .$$

But by Proposition 4.7, for $\Re(s) > -\frac{1}{2}$, $\mathsf{M}_{HP}^{s,N}$ is the unique invariant probability measure of (4.6) and so,

$$\int_0^\infty \boldsymbol{M}_u^{(-\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(-\nu)}\right)^\dagger \text{ is distributed as } \mathsf{M}_{HP}^{s,N}.$$

□

4.3 Proofs of auxiliary results

Proof of Proposition 4.6. The fact that (4.6) has a unique strong solution has been proven in Section 3.8 of Chapter 3 (by a standard argument found also in [41] and [60] for example). It suffices to check that,

$$\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} \left[\boldsymbol{X}_0 + \int_0^t \boldsymbol{M}_u^{(\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(\nu)}\right)^\dagger \right] \left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger}$$

indeed solves (4.6) for $\nu = \Re(s) + \frac{N}{2}, \mu = \sqrt{2}\Im(s)$. The initial condition is immediate and in order to ease notation, we will suppress any dependence on it in what follows. Let $\tilde{\boldsymbol{X}}_t^{\mu,\nu}$ denote the expression above. Then, applying Itô's formula we get,

$$\begin{aligned}
d\tilde{\boldsymbol{X}}_t^{\mu,\nu} &= d\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1}\right) \left[\boldsymbol{X}_0 + \int_0^t \boldsymbol{M}_u^{(\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(\nu)}\right)^\dagger \right] \left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger} \\
&+ \left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} \left[\boldsymbol{X}_0 + \int_0^t \boldsymbol{M}_u^{(\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(\nu)}\right)^\dagger \right] d\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger}\right) \\
&+ \left(\boldsymbol{M}_t^{(\nu)}\right)^{-1} \left[\boldsymbol{M}_t^{(\nu)} \left(\frac{d\boldsymbol{B}_t^{(\mu)} + d\left(\boldsymbol{B}_t^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_t^{(\nu)}\right)^\dagger \right] \left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger} \\
&+ d\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1}\right) \left[\boldsymbol{X}_0 + \int_0^t \boldsymbol{M}_u^{(\nu)} \left(\frac{d\boldsymbol{B}_u^{(\mu)} + d\left(\boldsymbol{B}_u^{(\mu)}\right)^\dagger}{\sqrt{2}} \right) \left(\boldsymbol{M}_u^{(\nu)}\right)^\dagger \right] d\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger}\right).
\end{aligned}$$

Note that, the terms of the form,

$$d\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1}\right)\left[\boldsymbol{M}_t^{(\nu)}\left(\frac{d\boldsymbol{B}_t^{(\mu)}+d\left(\boldsymbol{B}_t^{(\mu)}\right)^{\dagger}}{\sqrt{2}}\right)\left(\boldsymbol{M}_t^{(\nu)}\right)^{\dagger}\right]\left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger}=0,$$

$$\left(\boldsymbol{M}_t^{(\nu)}\right)^{-1}\left[\boldsymbol{M}_t^{(\nu)}\left(\frac{d\boldsymbol{B}_t^{(\mu)}+d\left(\boldsymbol{B}_t^{(\mu)}\right)^{\dagger}}{\sqrt{2}}\right)\left(\boldsymbol{M}_t^{(\nu)}\right)^{\dagger}\right]d\left(\left(\boldsymbol{M}_t^{(\nu)}\right)^{-\dagger}\right)=0,$$

by independence of $\boldsymbol{B}$ and the driving Brownian motion $\boldsymbol{W}$ of $\boldsymbol{M}^{(\nu)}$. Moreover, using the fact that for a (scalar) complex Brownian motion β we have the following quadratic covariation rules: $d\beta d\beta = 0, d\beta d\bar{\beta} = 2$; we easily obtain (we will do a similar and more complicated calculation below) for a matrix $\boldsymbol{A}$ and matricial complex Browian motion $\boldsymbol{W}$,

$$d\boldsymbol{W}_t\boldsymbol{A}d\boldsymbol{W}_t^{\dagger}=2\mathrm{Tr}(\boldsymbol{A})\boldsymbol{I}dt.$$

Hence,

$$\begin{aligned}
d\tilde{\boldsymbol{X}}_t^{\mu,\nu} &= -\frac{1}{\sqrt{2}}d\boldsymbol{W}_t\tilde{\boldsymbol{X}}_t^{\mu,\nu}+\frac{d\boldsymbol{B}_t}{\sqrt{2}}-\frac{1}{\sqrt{2}}\tilde{\boldsymbol{X}}_t^{\mu,\nu}d\boldsymbol{W}_t^{\dagger}+\frac{d\boldsymbol{B}_t^{\dagger}}{\sqrt{2}}-2\nu\tilde{\boldsymbol{X}}_t^{\mu,\nu}dt+\frac{2}{\sqrt{2}}\mu\boldsymbol{I}dt+\mathrm{Tr}(\tilde{\boldsymbol{X}}_t^{\mu,\nu})\boldsymbol{I}dt\\
&=\left[-\frac{1}{\sqrt{2}}d\boldsymbol{W}_t\tilde{\boldsymbol{X}}_t^{\mu,\nu}+\frac{d\boldsymbol{B}_t}{\sqrt{2}}\right]\left(\frac{\boldsymbol{I}+\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2}{2}\right)^{-\frac{1}{2}}\sqrt{\frac{\boldsymbol{I}+\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2}{2}}+\sqrt{\frac{\boldsymbol{I}+\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2}{2}}\left(\frac{\boldsymbol{I}+\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2}{2}\right)^{-\frac{1}{2}}\left[-\frac{1}{\sqrt{2}}d\boldsymbol{W}_t\tilde{\boldsymbol{X}}_t^{\mu,\nu}+\frac{d\boldsymbol{B}_t}{\sqrt{2}}\right]\\
&-2\nu\tilde{\boldsymbol{X}}_t^{\mu,\nu}dt+\frac{2}{\sqrt{2}}\mu\boldsymbol{I}dt+\mathrm{Tr}(\tilde{\boldsymbol{X}}_t^{\mu,\nu})\boldsymbol{I}dt.
\end{aligned}$$

Writing, $d\boldsymbol{\Gamma}_t=\left[-\frac{1}{\sqrt{2}}d\boldsymbol{W}_t\tilde{\boldsymbol{X}}_t^{\mu,\nu}+\frac{d\boldsymbol{B}_t}{\sqrt{2}}\right]\left(\frac{\boldsymbol{I}+\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2}{2}\right)^{-\frac{1}{2}}$ and then using Levy's characterization and $(d\boldsymbol{\Gamma}_t)_{ij}(d\boldsymbol{\Gamma}_t)_{i'j'}=0$, $(d\boldsymbol{\Gamma}_t)_{ij}(d\bar{\boldsymbol{\Gamma}}_t)_{i'j'}=2\delta_{i,i'}\delta_{j,j'}dt$ we deduce that $(\boldsymbol{\Gamma}_t;t\geq 0)$ is a complex Brownian matrix. The fact that $(d\boldsymbol{\Gamma}_t)_{ij}(d\boldsymbol{\Gamma}_t)_{i'j'}=0$ is immediate; to check $(d\boldsymbol{\Gamma}_t)_{ij}(d\bar{\boldsymbol{\Gamma}}_t)_{i'j'}=2\delta_{i,i'}\delta_{j,j'}dt$, writing $\boldsymbol{Y}_t = \left(\boldsymbol{I} + \left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2\right)^{-\frac{1}{2}}$ we have,

$$
\begin{aligned}
(d\boldsymbol{\Gamma}_t)_{ij}(d\bar{\boldsymbol{\Gamma}}_t)_{i'j'} &= \left(\sum_{k,l} -d\boldsymbol{W}_t^{i,k}\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{kl}\boldsymbol{Y}_t^{lj} + \sum_l d\boldsymbol{B}_t^{il}\boldsymbol{Y}_t^{lj}\right)\left(\sum_{k',l'} -d\bar{\boldsymbol{W}}_t^{i',k'}\overline{\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{k'l'}}\bar{\boldsymbol{Y}}_t^{l'j'} + \sum_{l'} d\bar{\boldsymbol{B}}_t^{i'l'}\bar{\boldsymbol{Y}}_t^{l'j'}\right)\\
&= \left[2\delta_{i,i'}\sum_{k,k',l,l'}\delta_{k,k'}\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{kl}\boldsymbol{Y}_t^{lj}\overline{\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{k'l'}}\bar{\boldsymbol{Y}}_t^{l'j'} + 2\delta_{i,i'}\sum_{l,l'}\delta_{l,l'}\boldsymbol{Y}_t^{lj}\bar{\boldsymbol{Y}}_t^{l'j'}\right]dt\\
&= \left[2\delta_{i,i'}\sum_{l,l'}\left[\sum_k\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{kl}\overline{\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{kl'}}\right]\boldsymbol{Y}_t^{lj}\bar{\boldsymbol{Y}}_t^{l'j'} + 2\delta_{i,i'}\sum_{l,l'}\delta_{l,l'}\boldsymbol{Y}_t^{lj}\bar{\boldsymbol{Y}}_t^{l'j'}\right]dt\\
&= \left[2\delta_{i,i'}\sum_{l,l'}\left[\sum_k\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{kl}\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^{l'k}\right]\boldsymbol{Y}_t^{lj}\boldsymbol{Y}_t^{j'l'} + 2\delta_{i,i'}\sum_{l,l'}\delta_{l,l'}\boldsymbol{Y}_t^{lj}\boldsymbol{Y}_t^{j'l'}\right]dt\\
&= \left[2\delta_{i,i'}\sum_{l,l'}\boldsymbol{Y}_t^{j'l'}\left[\left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2_{l'l} + \boldsymbol{I}_{l'l}\right]\boldsymbol{Y}_t^{lj}\right]dt\\
&= \left[2\delta_{i,i'}\left(\boldsymbol{Y}_t\left(\boldsymbol{I} + \left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2\right)\boldsymbol{Y}_t\right)_{j'j}\right]dt = 2\delta_{i,i'}\delta_{j,j'}dt,
\end{aligned}
$$

where we have used the fact that both $\tilde{\boldsymbol{X}}_t^{\mu,\nu}$ and $\boldsymbol{Y}_t$ are Hermitian in the fourth equality. Thus,

$$
d\tilde{\boldsymbol{X}}_t^{\mu,\nu} = d\boldsymbol{\Gamma}_t\sqrt{\frac{\boldsymbol{I} + \left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2}{2}} + \sqrt{\frac{\boldsymbol{I} + \left(\tilde{\boldsymbol{X}}_t^{\mu,\nu}\right)^2}{2}}d\boldsymbol{\Gamma}_t^\dagger - 2\nu\tilde{\boldsymbol{X}}_t^{\mu,\nu}dt + \frac{2}{\sqrt{2}}\mu\boldsymbol{I}dt + \mathrm{Tr}(\tilde{\boldsymbol{X}}_t^{\mu,\nu})\boldsymbol{I}dt.
$$

Finally, to match with (4.6), we just need to take $\nu = \Re(s) + \frac{N}{2}$, $\mu = \sqrt{2}\Im(s)$. $\square$

Proof of Proposition 4.7. This has already been observed in Section 3.8 of the previous chapter. The argument goes as follows. Let $\mathbb{U}(N)$ denote the $N\times N$ unitary group. Then, by $\mathbb{U}(N)$-invariance of the law of the dynamics of (4.6) (invariance under conjugation, $x \mapsto U^\dagger xU$, for $U \in \mathbb{U}(N)$), it suffices to show that its spectral evolution, denoted by $(x_1(t), \cdots x_N(t); t \geq 0)$ has $\mu_{HP}^{s,N}(dx)$ as its unique invariant probability measure. Using Theorem 4 of [77] for example, we obtain that $(x_1(t), \cdots x_N(t); t \geq 0)$ follows the stochastic differential system,

$$
dx_i(t) = \sqrt{2(1 + x_i^2(t))}d\beta_i(t) + \left(2\Im(s) + (2 - 2N - 2\Re(s))\,x_i(t) + \sum_{j\neq i}\frac{2\left(1 + x_i^2(t)\right)}{x_i(t) - x_j(t)}\right)dt,\ 1 \leq i \leq N,
$$

for some independent standard (real) Brownian motions $\{\beta_i\}_{i=1}^N$. It was proven in Lemma 3.12 of Chapter 3, using the general results of [78], that this system of SDEs has a unique strong solution, with no explosions or collisions, even if started from a degenerate point (when $x_i(0) = x_j(0)$ for $i \neq j$). Let $\left(P_{HP}^{s,N}(t); t \geq 0\right)$ denote the Markov semigroup associated with it. Then, checking invariance $\mu_{HP}^{s,N}P_{HP}^{s,N}(t) = \mu_{HP}^{s,N}$, $t \geq 0$ is particularly simple, since the argument becomes essentially one-dimensional. This is because the kernel, $P_{HP}^{s,N}(t)(x, dy)$ in

W^N, of the semigroup $\left(P_{HP}^{s,N}(t); t \geq 0\right)$ has a determinantal structure, given by an *h-transform* of a *Karlin-McGregor* semigroup. Namely,

$$P_{HP}^{s,N}(t)(x, dy) = e^{-ct} \frac{\Delta_N(x)}{\Delta_N(y)} \det\left(p_t^{s,N}(x_i, y_j)\right)_{i,j=1}^N dy_1 \cdots dy_N$$

where $x = (x_1, \cdots, x_N)$, $y = (y_1, \cdots, y_N)$, $p_t^{s,N}(z, w)$ is the strictly positive transition density, with respect to Lebesgue measure in $\mathbb{R}$, of the one-dimensional diffusion process with generator,

$$L_s^{(N)} = (w^2 + 1)\frac{d^2}{dw^2} + \left[(2 - 2N - 2\Re(s))\, w + 2\Im(s)\right] \frac{d}{dw},$$

which is furthermore, reversible with respect to the measure,

$$m_s^{(N)}(w)dw = (1 + w^2)^{-\Re(s)-N} e^{2\Im(s)\arg(1+iw)} dw$$

and finally c is a constant. Invariance and uniqueness of $\mu_{HP}^{s,N}$ then follow easily. The reader is referred to Proposition 3.14 of Chapter 3 for the details.

We can alternatively argue for uniqueness of the invariant measure $\mathsf{M}_{HP}^{s,N}(d\boldsymbol{X})$, by noting that the diffusion matrix of (4.6) is uniformly positive definite, from which we deduce (see [147] for example) that if $\mathsf{G}_{\boldsymbol{X}}$ denotes the generator of the unique solution of (4.6), then $\partial_t - \mathsf{G}_{\boldsymbol{X}}^*$ is hypoelliptic. □

Proof of Lemma 4.4. Let $T > 0$ be fixed. For $0 \leq t \leq T$, we have that,

$$\boldsymbol{M}_T^{(\nu)} = \boldsymbol{M}_{T-t}^{(\nu)} + \int_{T-t}^T \boldsymbol{M}_u^{(\nu)} d\boldsymbol{W}_u + \nu \int_{T-t}^T \boldsymbol{M}_u^{(\nu)} du.$$

Hence, by multiplying by $\left(\boldsymbol{M}_T^{(\nu)}\right)^{-1}$ and making the change of variables $u \mapsto T - u$ in the Lebesgue integral,

$$\begin{aligned}\boldsymbol{N}_t^{(\nu)} &= \boldsymbol{I} - \int_{T-t}^T \left(\boldsymbol{M}_T^{(\nu)}\right)^{-1} \boldsymbol{M}_u^{(\nu)} d\boldsymbol{W}_u - \nu \int_{T-t}^T \left(\boldsymbol{M}_T^{(\nu)}\right)^{-1} \boldsymbol{M}_u^{(\nu)} du \\ &= \boldsymbol{I} - \int_{T-t}^T \left(\boldsymbol{M}_T^{(\nu)}\right)^{-1} \boldsymbol{M}_u^{(\nu)} d\boldsymbol{W}_u - \nu \int_0^t \boldsymbol{N}_u^{(\nu)} du.\end{aligned}$$

Now, to treat the stochastic integral term, begin by writing $\tilde{\boldsymbol{W}}_t = \boldsymbol{W}_{T-t} - \boldsymbol{W}_T$ for the time-reversed Brownian motion. We note that, this is again a Brownian motion with filtration given by,

$$\mathcal{F}_{r,s}^{\tilde{\boldsymbol{W}}} = \sigma\left(\tilde{\boldsymbol{W}}_s - \tilde{\boldsymbol{W}}_r | r \leq s \leq T\right) = \sigma\left(\boldsymbol{W}_u | T - s \leq u \leq T - r\right) = \mathcal{F}_{T-s,T-r}^{\boldsymbol{W}}.$$

Using an approximation by Riemann sums, see for example Proposition 7.2.11 of [72] where this is done, we can write the stochastic integral in consideration as an Itô integral with respect to the time-reversed Brownian motion $\tilde{\boldsymbol{W}}$, namely,

$$\int_{T-t}^{T}\left(\boldsymbol{M}_T^{(\nu)}\right)^{-1}\boldsymbol{M}_u^{(\nu)}d\boldsymbol{W}_u=-\int_0^t\boldsymbol{N}_u^{(\nu)}d\tilde{\boldsymbol{W}}_u-\int_0^t d\left\langle\boldsymbol{N}^{(\nu)},\tilde{\boldsymbol{W}}\right\rangle_u.$$

Observe that, the martingale part of $d\boldsymbol{N}^{(\nu)}$ is $-\boldsymbol{N}^{(\nu)}d\tilde{\boldsymbol{W}}$ and thus,

$$\int_0^t d\left\langle\boldsymbol{N}^{(\nu)},\tilde{\boldsymbol{W}}\right\rangle_u=\int_0^t\boldsymbol{N}_u^{(\nu)}d\left\langle\tilde{\boldsymbol{W}},\tilde{\boldsymbol{W}}\right\rangle_u=0,$$

since we are dealing with complex Brownian motions (in case we were working with real Brownian matrices we would have picked up an extra drift term). The result then follows. □

Proof of Lemma 4.5. This is essentially an adaptation of Lemma 11 of [135]. We consider the following stochastic process $\left(\boldsymbol{Z}_t^{(-\nu)};t\geq 0\right)=\left(\boldsymbol{M}_t^{(-\nu)}\left(\boldsymbol{M}_t^{(-\nu)}\right)^{\dagger};t\geq 0\right)$. By developing $d\left(\boldsymbol{M}_t^{(-\nu)}\left(\boldsymbol{M}_t^{(-\nu)}\right)^{\dagger}\right)$ we get the following closed matrix SDE,

$$d\boldsymbol{Z}_t^{(-\nu)}=\frac{1}{\sqrt{2}}\sqrt{\boldsymbol{Z}_t^{(-\nu)}}d\boldsymbol{W}_t\sqrt{\boldsymbol{Z}_t^{(-\nu)}}+\frac{1}{\sqrt{2}}\sqrt{\boldsymbol{Z}_t^{(-\nu)}}d\boldsymbol{W}_t^{\dagger}\sqrt{\boldsymbol{Z}_t^{(-\nu)}}+(N-2\nu)\boldsymbol{Z}_t^{(-\nu)}dt,$$

for a complex matrix Brownian motion $(\boldsymbol{W}_t;t\geq 0)$. By Theorem 4 of [77] the eigenvalue evolution $\left(\eta_1^{(-\nu)}(t)\leq\cdots\leq\eta_N^{(-\nu)}(t);t\geq 0\right)$ of $\left(\boldsymbol{M}_t^{(-\nu)}\left(\boldsymbol{M}_t^{(-\nu)}\right)^{\dagger};t\geq 0\right)$, which form the squared singular values of $\left(\boldsymbol{M}_t^{(-\nu)};t\geq 0\right)$, satisfies,

$$d\eta_i^{(-\nu)}(t)=\sqrt{2}\eta_i^{(-\nu)}(t)d\beta_i(t)+\left[(N-2\nu)\,\eta_i^{(-\nu)}(t)+\sum_{k\neq i}\frac{2\eta_i^{(-\nu)}(t)\eta_k^{(-\nu)}(t)}{\eta_i^{(-\nu)}(t)-\eta_k^{(-\nu)}(t)}\right]dt,\ 1\leq i\leq N,$$

for some independent standard (real) Brownian motions $\{\beta_i\}_{i=1}^N$. Moreover, by making the change of variables $\delta_i^{(-\nu)}=\log\left(\eta_i^{(-\nu)}\right)$ we arrive at,

$$\begin{aligned}d\delta_i^{(-\nu)}(t)&=\sqrt{2}d\beta_i(t)+\left[(N-1-2\nu)+\sum_{k\neq i}\frac{2e^{\delta_k^{(-\nu)}(t)}}{e^{\delta_i^{(-\nu)}(t)}-e^{\delta_k^{(-\nu)}(t)}}\right]dt\\&=\sqrt{2}d\beta_i(t)+\left[-2\nu+\sum_{k\neq i}\frac{e^{\delta_i^{(-\nu)}(t)}+e^{\delta_k^{(-\nu)}(t)}}{e^{\delta_i^{(-\nu)}(t)}-e^{\delta_k^{(-\nu)}(t)}}\right]dt\ 1\leq i\leq N.\end{aligned}$$

As in the proof of Lemma 11 of [135], we observe the following: First,

$$\sum_{k\neq 1}\frac{e^{\delta_1^{(-\nu)}(t)}+e^{\delta_k^{(-\nu)}(t)}}{e^{\delta_1^{(-\nu)}(t)}-e^{\delta_k^{(-\nu)}(t)}}\leq 1-N$$

and furthermore, that changing i to $i+1$ the interaction term changes by at most,

$$2\frac{e^{\delta_{i+1}^{(-\nu)}(t)}+e^{\delta_{i}^{(-\nu)}(t)}}{e^{\delta_{i+1}^{(-\nu)}(t)}-e^{\delta_{i}^{(-\nu)}(t)}}.$$

Thus, for $i=1,\dots,N-1$, the difference $\delta_{i+1}^{(-\nu)}-\delta_{i}^{(-\nu)}$ is bounded above by the solution of,

$$dy_i(t) = \sqrt{2}\left(d\beta_{i+1}(t)-d\beta_i(t)\right)+2\left(\frac{1+e^{-y_i(t)}}{1-e^{-y_i(t)}}\right)dt$$

and similarly, $\delta_1^{(-\nu)}$ by the solution of,

$$d\tilde{\delta}_1^{(-\nu)}(t) = \sqrt{2}d\beta_1(t)+(-2\nu+1-N)dt.$$

Hence,

$$\lim_{t\to\infty}\frac{\delta_N^{(-\nu)}(t)}{t}\le\lim_{t\to\infty}\frac{\tilde{\delta}_1^{(-\nu)}(t)}{t}+\sum_{i=1}^{N-1}\lim_{t\to\infty}\frac{y_i(t)}{t}=(-2\nu+1-N)+2(N-1)\text{ almost surely.}$$

□

Chapter 5

Random surface growth and Karlin-McGregor polynomials

5.1 Introduction

5.1.1 Determinantal structures in inhomogeneous random growth models

This chapter revolves around two sets of closely related problems and ideas. One of them is, the construction of consistent dynamics on the levels of certain branching graphs and the other is, the exact computation of correlations in random stepped surface growth processes. The first part can also be seen as the discrete analogue of the work done in Chapter 1.

These probabilistic models can be viewed as dynamics on (discrete) interlacing arrays, namely multilevel configurations of particles that satisfy some constraints (that we make precise below), see Figure 1 below for an illustration. Such (2+1)-dimensional dynamics (2 space and 1 time dimensions) have been extensively studied in the past decade, see [21],[23],[47],[19],[33],[34] (see also Chapter 1 for the continuum setting). In addition to being interesting in its own right a further motivation for this study is the following phenomenon: the exact solvability of a wide class of (1+1)-dimensional models such as the Totally Asymmetric Simple Exclusion Process (TASEP) is a by product of the fact that they appear as projections of these higher dimensional models, see [21].

In many of these papers (see [21], [23], [47]) the models considered give rise to determinantal point processes: for a point process on a discrete space $\mathfrak{X}$ we say that it is determinantal if for all $n \geq 1$ its correlation functions ρ_n are given as determinants of a two variable function $\mathsf{K} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$:

$$\rho_n\left(\mathfrak{x}_1, \cdots, \mathfrak{x}_n\right) = \det\left[\mathsf{K}\left(\mathfrak{x}_i, \mathfrak{x}_j\right)\right]_{i,j=1}^{n}.$$

Thus, all probabilistic information about the model is encoded in the function K and questions about its limit behaviour reduce to asymptotic analysis of K.

In all aforementioned papers, the jump rates of particles on each level had a rather special algebraic dependence on their positions. The main novelty of our contribution is that we allow (essentially) arbitrary jump rates for individual particles depending on the position in the horizontal direction, while retaining the determinantal point process structure.

For many of the works in Integrable Probability the exact solvability of the models can be traced down to a rich duality structure, see [20], [100], [101]. In this chapter a key role is played by the famous Siegmund duality for birth and death chains (the discrete analogue of the duality between diffusions in chapter 1), going back to Karlin and McGregor, see [92], [93]: Consider a birth and death chain in $I = \mathbb{N}$ (reflecting at 0) or a bilateral chain in $I = \mathbb{Z}$ with generator $\mathcal{D}$ given by the birth rates $\lambda(x)$ and death rates $\mu(x)$ (the positive functions $\lambda(\cdot), \mu(\cdot)$ can be essentially arbitrary modulo technicalities). Then we define its Siegmund dual (which is absorbed at -1 in the birth and death chain case) with generator $\hat{\mathcal{D}}$ and birth rates given by $\hat{\lambda}(x) = \mu(x+1)$ and death rates by $\hat{\mu}(x) = \lambda(x)$. The key property these dual chains satisfy is the following: if we consider two copies $X(t)$ and $\hat{X}(t)$ with generators $\mathcal{D}$ and $\hat{\mathcal{D}}$ respectively then for $x, y \in I$ and $t \geq 0$ we have:

$$\mathbb{P}_x \left(X(t) \leq y \right) = \mathbb{P}_y \left(\hat{X}(t) \geq x \right).$$

Then, from considering a coalescing flow of birth and death chains we obtain an explicit formula in terms of block determinants, describing a joint evolution (X, Y) of interacting $\mathcal{D}$ and $\hat{\mathcal{D}}$-chains. To explain this further we need some notation. Let us denote the n-dimensional (discrete) Weyl chamber, where all the x_i are either in $\mathbb{N}$ or $\mathbb{Z}$, by

$$W^n = \{(x_1, \cdots, x_n) : x_1 < \cdots < x_n\}.$$

Then, for $x \in W^{n+1}$ and $y \in W^n$ we will say that x and y *interlace* and write $y \prec x$ if (note the position of $<$ and $\leq$):

$$x_1 \leq y_1 < x_2 \leq \cdots \leq y_n < x_{n+1}$$

and denote by $W^{n,n+1}$ the space of such pairs (x, y).

The joint evolution (X, Y) takes values in $W^{n,n+1}$ and can be described as follows: Y is autonomous and evolves as n $\hat{\mathcal{D}}$-chains conditioned not to intersect by a Doob's h-transform and X as $n+1$ $\mathcal{D}$-chains *pushed* and *blocked* by the Y-particles, when the process is on the boundary of $W^{n,n+1}$ (the interactions are local), in order for the interlacing to remain true. In particular, the X-particles never intersect. As a by product of the special structure of these formulae, we obtain as part of our first set of results, that under special initial conditions of (X, Y) the non-autonomous X-component is in fact distributed as a Markov chain. Its evolution being that of $n+1$ $\mathcal{D}$-chains conditioned not to intersect by an explicit Doob's transformation, given in terms of the original transform of the Y-component.

Analogous formulae, having essentially the same structure, are also obtained for (X, Y) taking values in $W^{n,n}$ given by interlacing sequences of the form $y_1 \leq x_1 < y_2 \leq \cdots \leq x_n$ (again we write $y \prec x$). It is then possible to concatenate such two-level couplings in a consistent fashion, to build a multilevel process with interlacing components such that, if started according to certain initial conditions each level evolves as a Markov chain in its own right with an explicit distribution.

Then, we go on to consider a particular choice of such consistent multilevel dynamics, that we call the alternating construction. Its distribution at time $t \geq 0$ gives rise to a determinantal point process. To compute its correlation kernel explicitly we make heavy use of the spectral theory for birth and death chains and their associated orthogonal polynomials, developed by Karlin and McGregor in [92], [93].

We now proceed to explain our main results in detail and how they relate to other works in the field of Integrable Probability.

5.1.2 Intertwinings and consistent multilevel dynamics

We make precise our first set of results. To begin, we need some definitions. We write $p_t(x, y)$ for the transition density of the $\mathcal{D}$-chain and $\pi(\cdot)$ for the measure with respect to which it is reversible. Similarly we write $\hat{p}_t(x, y)$ and $\hat{\pi}(\cdot)$ for the ones associated to its Siegmund dual, the $\hat{\mathcal{D}}$-chain. We shall denote the Karlin-McGregor semigroup associated to n $\mathcal{D}$-chains killed when they intersect by $\left(P^n_t; t \geq 0\right)$. This is given by the determinantal transition kernel:

$$p^n_t(x, y) = \det(p_t(x_i, y_j))^n_{i,j=1}.$$

Similarly, we will write $\left(\hat{P}^n_t; t \geq 0\right)$ for the one associated to n $\hat{\mathcal{D}}$-chains. We also define the positive kernels:

$$(\Lambda_{n,n+1} f)(x) = \sum_{y \prec x} \prod_{i=1}^{n} \hat{\pi}(y_i) f(y), \quad (\Lambda_{n,n} f)(x) = \sum_{y \prec x} \prod_{i=1}^{n} \pi(y_i) f(y).$$

Then, we have the following Theorem, proven as part of more general results in Section 5.2.3 (for the shortest path to a proof of this particular statement see Remark 5.21).

Theorem 5.1. *For $t \geq 0$:*

$$P^{n+1}_t \Lambda_{n,n+1} = \Lambda_{n,n+1} \hat{P}^n_t, \tag{5.1}$$

$$\hat{P}^n_t \Lambda_{n,n} = \Lambda_{n,n} P^n_t. \tag{5.2}$$

After a Doob's h-transformation (see Section 5.2.3), by a strictly positive eigenfunction $h(\cdot)$ of either $\hat{P}^n_t$ or P^n_t, the relations above take the form:

$$P_{N+1}(t) \Lambda^{N+1}_N = \Lambda^{N+1}_N P_N(t) \ , \ \forall t \geq 0. \tag{5.3}$$

Here, the semigroups $P_N(t), P_{N+1}(t)$ are Markov on $W^{\mathsf{n}(N)}$ and $W^{\mathsf{n}(N+1)}$ respectively, where $\mathsf{n}(N+1) \in \{\mathsf{n}(N), \mathsf{n}(N)+1\}$. Moreover, Λ_N^{N+1} is a Markov kernel from $W^{\mathsf{n}(N+1)}$ to $W^{\mathsf{n}(N)}$.

As mentioned in the subsection above, these results are a by product of a two-level coupling of interacting $\mathcal{D}$-chains and $\hat{\mathcal{D}}$-chains, coming from considering a coalescing stochastic flow, which remarkably admits an explicit transition kernel in terms of a block determinant. Although, aspects of this probabilistic argument appeared in the seminal work of Warren [164] in the context of the Brownian web or Arratia flow (see also Subsection 1.2.3 of Chapter 1), our exposition in Section 5.2.2 is new, being both elementary and completely self-contained.

Branching graphs Sequences of stochastic evolutions satisfying (5.3) can be recast in the framework of coherent dynamics on branching graphs. Let us briefly and informally describe this, all notions are made precise in Section 5.4. We consider a graded graph Γ, with vertex set $\sqcup_N V_N$ such that $V_N = W^{\mathsf{n}(N)}$, where $\mathsf{n}(1) \le \mathsf{n}(2) \le \cdots, \mathsf{n}(i+1) - \mathsf{n}(i) \in \{0,1\}$. Two vertices $y \in V_N$ and $x \in V_{N+1}$ are connected by an edge if and only if x and y interlace (more precisely if $y \prec x$). We assign certain multiplicities (positive weights) to each edge and from this, see Section 5.4, we can associate for all N a natural Markov kernel Λ_N^{N+1} from V_{N+1} to V_N.

The semigroups $P_N(t)$ can be viewed as dynamics on the individual levels V_N of Γ. We will moreover say that they are coherent with respect to the Λ_N^{N+1} if (5.3) holds.

A motivation for studying such relations comes from the method of intertwiners of Borodin and Olshanski, that we already made essential use of in Chapter 3: it takes as input a tower of relations (5.3) for all N and produces a Markov process with semigroup $P_\infty(t)$ on the boundary Ω_Γ of the graph Γ. Informally the boundary Ω_Γ of Γ is the space parametrizing extremal coherent sequences of probability measures: $\{\mu_N\}_{N\ge1}$ on $\{V_N\}_{N\ge1}$ that satisfy,

$$\mu_{N+1}\Lambda_N^{N+1} = \mu_N$$

and that cannot be decomposed into convex combinations of other such sequences.

The method was applied in the context of two well known branching graphs related to representation theory, the Gelfand-Tsetlin graph in [28] by Borodin and Olshanski and the type-BC graph in [50] by Cuenca. In Section 5.5 we give alternative proofs of their main results, showing how they follow from Theorem 5.1. For a brief comparison between the proofs see Remark 5.21.

Push-Block dynamics Now, suppose we are given a sequence of processes with semigroups $\{P_n(t)\}_{n=1}^N$ and Markov kernels $\{\Lambda_n^{n+1}\}_{n=1}^{N-1}$ satisfying (5.3) derived from Theorem 5.1. We then construct a multilevel process $\left(\left(X^1(t), \cdots, X^N(t)\right); t \ge 0\right)$, taking values in $W^{\mathsf{n}(1)} \times \cdots \times W^{\mathsf{n}(N)}$ so that consecutive levels interlace and which satisfies a Gibbs property with respect to the Markov kernels Λ_n^{n+1}. The interactions between particles (coordinates

X_i^n) are through the so called push-block dynamics.

These are described informally as follows (see Section 5.3.2 for the rigorous description): Each particle has two independent exponential clocks with (not necessarily equal) rates depending only on its position (and not of other particles) for jumping to the right and to the left respectively by one. Suppose the clock for jumping to the right of particle X_i^n at level n rings first. Then the particle will attempt to jump to the right by one; if the interlacing with level $n-1$ is no longer satisfied this jump is not allowed and we say the particle is blocked. Otherwise, it moves by one to the right, possibly triggering some pushing moves. Namely, if the interlacing is no longer preserved with the next level then the particle at level $n+1$ with respect to which the interlacing is broken also moves (instantaneously) to the right by one. This pushing is propagated to higher levels.

Then, in Section 5.3.3 we prove a result of the following sort, that we state informally here (see Propositions 5.28 and 5.30 for the precise statements):

Proposition 5.2 (Informal statement)**.** *Suppose the Markov kernels $\{\Lambda_n^{n+1}\}_{n=1}^{N-1}$ and semigroups $(P_n(t); t \geq 0)_{n=1}^{N}$ obtained from Theorem 5.1 satisfy the intertwining relations (5.3) for $n = 1, \cdots, N-1$. Then, there exists a Markovian coupling $\left(\left(X^1(t), \cdots, X^N(t)\right); t \geq 0\right)$ (taking values in an interlacing array) evolving according to push-block dynamics with explicit rates (see Subsections 5.3.2 and 5.3.3) such that the following hold: Assume the process $\left(\left(X^1(t), \cdots, X^N(t)\right); t \geq 0\right)$ is initialized according to the Gibbs measure, where $\mathfrak{M}_N(\cdot)$ is an arbitrary probability measure on $\mathsf{W}^{\mathsf{n}(N)}$:*

$$\mathsf{Prob}_{1,\cdots,N}^{\mathfrak{M}_N}\left(x^1, \cdots, x^N\right) = \mathfrak{M}_N(x^N)\Lambda_{N-1}^N\left(x^N, x^{N-1}\right)\cdots\Lambda_1^2\left(x^2, x^1\right).$$

Then, the distribution at time T of $\left(X^1(T), \cdots, X^N(T)\right)$ is given by the evolved Gibbs measure:

$$\mathsf{Prob}_{1,\cdots,N}^{\mathfrak{M}_N P_N(T)}\left(x^1, \cdots, x^N\right) = \left[\mathfrak{M}_N P_N(T)\right](x^N)\Lambda_{N-1}^N\left(x^N, x^{N-1}\right)\cdots\Lambda_1^2\left(x^2, x^1\right).$$

Moreover, for each $n = 1, \cdots, N$ the projection to $(X^n(t); t \geq 0)$ is a Markov process evolving according to $(P_n(t); t \geq 0)$.

By the special structure of the Markov kernels and semigroups involved (for certain initial measures $\mathfrak{M}_N$) we get that the evolved measure $\mathsf{Prob}_{1,\cdots,N}^{\mathfrak{M}_N P_N(T)}\left(x^1, \cdots, x^N\right)$ is given as a certain product of determinants. Such measures, by the celebrated Eynard-Mehta Theorem (see [35]), give rise to determinantal point processes with an extended correlation kernel K, which can in principle be computed (see Remark 5.29).

5.1.3 Alternating construction

We will now consider in some detail a particular choice of consistent dynamics. These dynamics give rise via Proposition 5.2 to an interacting interlacing particle system with a wall. Such a system can be mapped to a random growth and decay model of a stepped surface under a certain correspondence between particles and lozenges/cubes, see Figure 5.1 for an illustration. We first need a definition. Denote by $\mathbb{GT}_{\mathbf{s}}(\infty)$ the set of infinite symplectic Gelfand-Tsetlin patterns, interlacing arrays of the form, where all particles live in $\mathbb{N}$, with the *origin* playing the role of a wall,

$$\mathbb{GT}_{\mathsf{s}}(\infty)=\left\{\mathbb{X}=\left(\mathbb{X}^{(0,1)},\mathbb{X}^{(1,1)},\mathbb{X}^{(1,2)},\cdots\right):\mathbb{X}^{(i,i)}\in W^i,\mathbb{X}^{(i,i+1)}\in W^{i+1};\mathbb{X}^{(i-1,i)}\prec\mathbb{X}^{(i,i)}\prec\mathbb{X}^{(i,i+1)}\right\}.$$

The dynamics go as follows: Particles at level $\mathbb{X}^{(i,i+1)}$ evolve as $i+1$ independent $\mathcal{D}$-chains which are pushed and blocked by particles at level $\mathbb{X}^{(i,i)}$, which themselves evolve as i independent $\hat{\mathcal{D}}$-chains that are in turn pushed and blocked by particles at level $\mathbb{X}^{(i-1,i)}$ and so forth, see Figure 5.1 for an example. We call this the *alternating construction*, since we alternate between using the jump rates for $\mathcal{D}$ and $\hat{\mathcal{D}}$-chains on odd and even levels. We think of the position-dependent jump, equivalently growth and decay, rates as inhomogeneities of the surface.

To make the connection with Proposition 5.2, the projection on level $\mathbb{X}^{(n,n+1)}$ evolves according to the semigroup with transition kernel:

$$\frac{h_{n,n+1}(y_1,\cdots,y_{n+1})}{h_{n,n+1}(x_1,\cdots,x_{n+1})}\det(p_t(x_i,y_j))_{i,j=1}^{n+1}$$

and on level $\mathbb{X}^{(n,n)}$ according to:

$$\frac{h_{n,n}(y_1,\cdots,y_n)}{h_{n,n}(x_1,\cdots,x_n)}\det(\hat{p}_t(x_i,y_j))_{i,j=1}^{n}.$$

Moreover, the harmonic functions $h_{n,n+1}(\cdot)$ and $h_{n,n}(\cdot)$ are given by:

$$h_{n,n+1}(\cdot)=(\Lambda_{n,n+1}\Lambda_{n,n}\cdots\Lambda_{1,1}\mathbf{1})(\cdot)\ ,\ h_{n,n}(\cdot)=(\Lambda_{n,n}\Lambda_{n-1,n}\cdots\Lambda_{1,1}\mathbf{1})(\cdot).$$

The distribution of this particle system at time t determines a point process denoted by Ξ^t. Assume that all particles are initially *fully packed* i.e. at levels $(i-1,i)$ and (i,i) we have our i particles at positions $0<1<2<\cdots<i-1$ (see Figure 5.1). We shall also use the following notation throughout: the variable $z=((n_1,n_2),x)$ will denote the level (n_1,n_2) and (horizontal) position x of the particle.

We now explain how the model in the seminal work of Borodin and Kuan [23] related to the representation theory of the infinite dimensional orthogonal group $O(\infty)$ and its recent generalization by Cerenzia and Kuan [47] are special cases of this construction: they simply correspond to a particular choice of the rate functions $\lambda(\cdot),\mu(\cdot)$. The rates considered by Cerenzia and Kuan in [47], depending on two real parameters $\alpha,\beta>-1$, are the following:

$$\lambda(n)=\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\frac{2(n+\alpha+1)}{2n+\alpha+\beta+2},$$
$$\mu(n)=\frac{n+\beta}{2n+\alpha+\beta}\frac{2n}{2n+\alpha+\beta+1}.$$

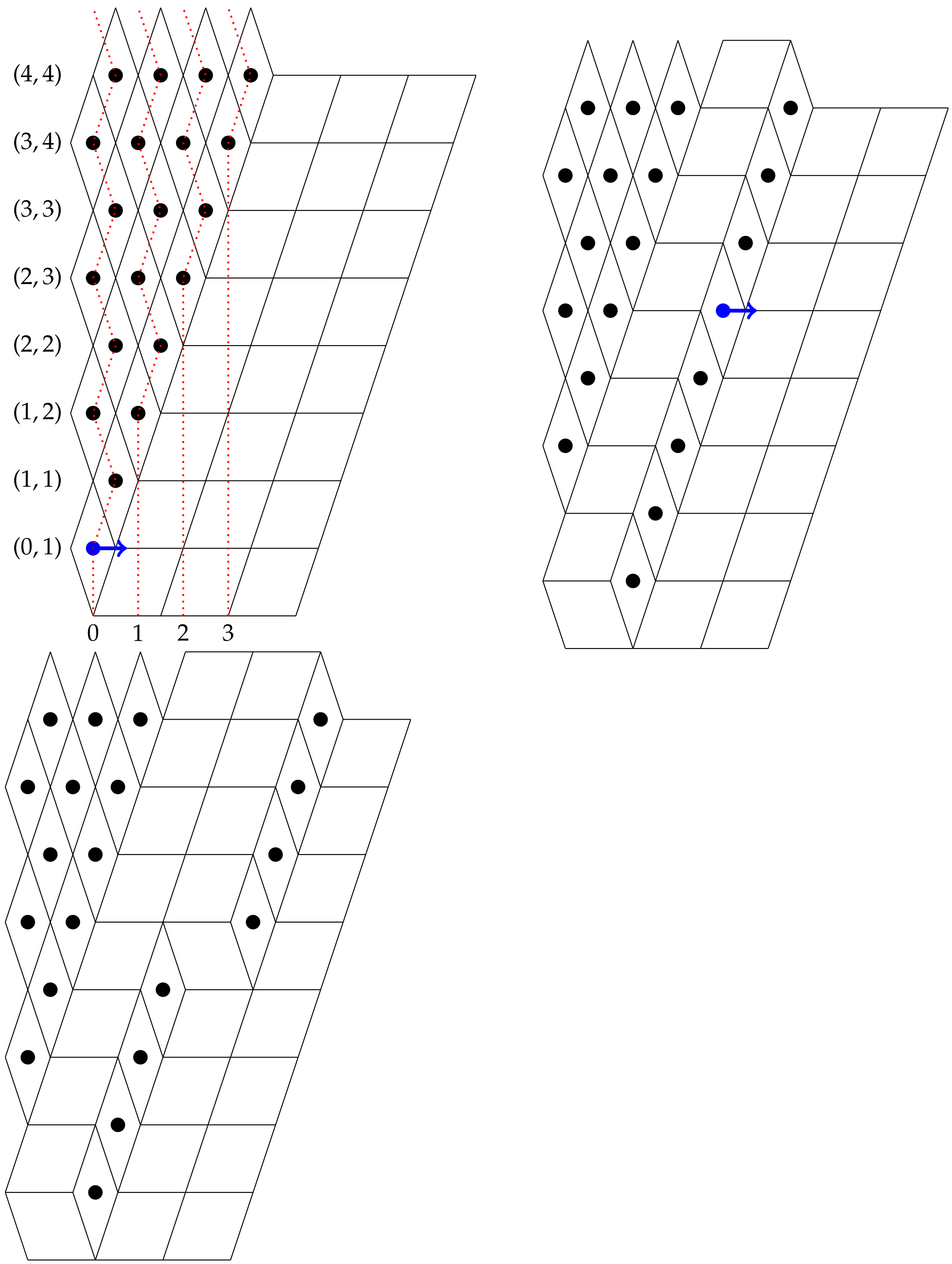


Figure 5.1: The visualisation of a particle configuration of $\mathbb{GT}_{\mathbf{s}}(\infty)$ as a stepped surface. In the first figure the fully packed initial condition is depicted. Particle $x_1^{(0,1)}$ wants to jump to the right and in doing so, pushes all the particles indexed $x_i^{(i-1,i)}$ and $x_i^{(i,i)}$ to the right by one as well, resulting in the surface shown in the second figure. Next, particle $x_3^{(2,3)}$ jumps to the right by one and this produces the stepped surface of the last figure.

For $\alpha = \beta = -\frac{1}{2}$ these specialize to the model studied by Borodin and Kuan in [23] while for $-\alpha = \beta = \frac{1}{2}$ they specialize to the model studied by Cerenzia [46] related to the infinite dimensional symplectic group $Sp(\infty)$.

Finally, as explained in the previous subsection, the evolved Gibbs measures for these dynamics are given as products of determinants and by making use of (one of the many variants of) the famous Eynard-Mehta Theorem (see [35]) it is standard that there is an underlying determinantal structure for this point process. However, to compute the correlation kernel $\mathcal{K}^t$ explicitly one needs to either invert a Gram matrix or solve a biorthogonalization problem, which is usually a formidable task.

5.1.4 Explicit computation of correlation kernel and scaling limit

It is at this point that a further insight is required in order to proceed. We make use of the spectral theory for one-dimensional birth and death chains first developed by Karlin and McGregor in [92] and [93]. More precisely we define the polynomials $Q_i(x)$ through the three term recurrence:

$$Q_0(x) = 1, -xQ_0(x) = -(\lambda(0) + \mu(0))Q_0(x) + \lambda(0)Q_1(x),$$
$$-xQ_n(x) = \mu(n)Q_{n-1}(x) - (\lambda(n) + \mu(n))Q_n(x) + \lambda(n)Q_{n+1}(x).$$

These are orthogonal with respect to the *spectral measure* $\mathrm{d}\mathfrak{w}(x)$ on $\mathbb{R}_+$ with support $\mathfrak{I}$,

$$\int_0^\infty Q_i(x)Q_j(x)\mathrm{d}\mathfrak{w}(x) = \frac{1}{\pi(j)}\delta_{ij}.$$

If we view $\mathcal{D}_k$, the generator of the birth and death chain with rates $(\lambda(\cdot), \mu(\cdot))$, as a difference operator in the discrete variable k, then the three term recurrence takes the form of an eigenfunction relation, with eigenvalue $x \geq 0$:

$$\mathcal{D}_k Q_k(x) = -xQ_k(x).$$

These ingredients provide the following spectral expansion for the transition density of the chain:

$$p_t(i, j) = \pi(j)\int_0^\infty e^{-tx}Q_i(x)Q_j(x)\mathrm{d}\mathfrak{w}(x).$$

One can also define the polynomials $\hat{Q}_k$ and measure $\hat{\mathfrak{w}}$ associated to the Siegmund dual chain and many relations exist between these dual polynomials, which can be found in Section 5.6.

We then go on to introduce and study in detail, from a probabilistic perspective in Sections 5.7 to 5.9 (see also Subsection 5.1.5 below), their multivariate versions: For $\nu \in W^n$, we consider the n-variate polynomials given by, with $x=(x_1,\cdots,x_n)$ in $\mathbb{R}^n$,

$$\mathfrak{Q}_\nu(x)=\frac{\det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^n}{\det\left(x_j^{i-1}\right)_{i,j=1}^n},\quad \hat{\mathfrak{Q}}_\nu(x)=\frac{\det\left(\hat{Q}_{\nu_i}(x_j)\right)_{i,j=1}^n}{\det\left(x_j^{i-1}\right)_{i,j=1}^n}.$$

We call these the Karlin-McGregor polynomials, since they were first introduced by Karlin and McGregor, in their original study of intersection probabilities of birth and death chains in [94]. We can obtain the Markov kernels associated to the alternating construction from branching rules of these multivariate polynomials (see Section 5.7, also Remark 5.72). In particular the harmonic functions $h_{n,n+1}(\nu)$ and $h_{n,n}(\nu)$ are given by:

$$h_{n,n+1}(\nu)=(-1)^{\binom{n}{2}}\lambda_0^{\binom{n}{2}}\mathfrak{Q}_\nu(\vec{0})\ ,\ h_{n,n}(\nu)=(-1)^{\binom{n-1}{2}}\lambda_0^{\binom{n-1}{2}}\hat{\mathfrak{Q}}_\nu(\vec{0}).$$

Moving on, it is only after expressing the entries of the determinants appearing in the distribution of the growth process starting from the fully packed initial condition in terms of these one dimensional orthogonal polynomials and the spectral measures, that it is possible to see/guess what the solution to the biorthogonalization problem is. Then we proceed to carefully check that it is indeed the solution. All of this is done in detail in Section 5.10. Finally, after some more algebraic manipulations we arrive at the following result, proven as a special case of the more general Theorem 5.69 in the text:

Theorem 5.3. *Let $\mathfrak{I}$ be compact then the correlation functions $\{\rho_k^t\}_{k\ge 0}$ of the point process Ξ^t, associated to the alternating construction starting from the fully packed initial condition, are determinantal:*

$$\rho_k^t(z_1,\cdots,z_k)\overset{\text{def}}{=}\Xi^t(\{E\in\mathbb{GT}_{\mathbf{s}}(\infty)\text{ s.t. }\{z_1,\cdots,z_k\}\subset E\})=\det\left(\mathcal{K}^t(z_i,z_j)\right)_{i,j=1}^k \tag{5.4}$$

where $\mathcal{K}^t$ is given by,

$$\begin{aligned}\mathcal{K}^t\left(((n_1,n_2),i),(m_1,m_2),j))\right)=\frac{1}{2\pi\mathrm{i}}\oint_{u\in\mathsf{C}(\mathfrak{I})}\int_{x\in\mathfrak{I}}\tilde{\mathcal{P}}_j(u)\bar{\mathcal{P}}_i(x)\frac{x^{n_2}}{u^{m_2}}\frac{e^{-tx}}{(x-u)e^{-tu}}d\mathfrak{m}(x)du\\+\mathbf{1}\left((n_1,n_2)\ge(m_1,m_2)\right)\int_{\mathfrak{I}}\bar{\mathcal{P}}_i(x)x^{n_2-m_2}\tilde{\mathcal{P}}_j(x)d\mathfrak{m}(x)\end{aligned} \tag{5.5}$$

and,

$$(\bar{\mathcal{P}},\tilde{\mathcal{P}},\mathfrak{m})=\begin{cases}(\pi_iQ_i,Q_j,\mathfrak{w})\text{ if }(n_1,n_2),(m_1,m_2)=(n,n+1),(m,m+1)\\(\pi_iQ_i,\hat{Q}_j,\mathfrak{w})\text{ if }(n_1,n_2),(m_1,m_2)=(n,n+1),(m,m)\\(\hat{\pi}_i\hat{Q}_i,Q_j,\hat{\mathfrak{w}})\text{ if }(n_1,n_2),(m_1,m_2)=(n,n),(m,m+1)\\(\hat{\pi}_i\hat{Q}_i,\hat{Q}_j,\hat{\mathfrak{w}})\text{ if }(n_1,n_2),(m_1,m_2)=(n,n),(m,m)\end{cases}. \tag{5.6}$$

The contour $\mathsf{C}(\mathfrak{I})$ is positively oriented and encircles the support $\mathfrak{I}$ and 0.

That the interval of orthogonality $\mathfrak{I}$ needs to be compact is a technical analytic requirement and we indicate in several remarks how this can be removed. In the model of Cerenzia and Kuan [47] mentioned above the $Q_i(x)$ are the Jacobi polynomials, which specialize to the Chebyshev polynomials of the earlier works [23], [46].

We then go on to consider a particular scaling limit, at a finite distance from the wall and make the connection with Borodin and Olshanski's work in [30] on discrete determinantal ensembles associated to continuous orthogonal polynomials.

More precisely, suppose we scale time as $t(N) = N\tau$ and the arguments of the kernel as $(\tilde{m}_1(N), \tilde{m}_2(N)) = (\lfloor N\eta \rfloor + m_1, \lfloor N\eta \rfloor + m_2)$ and $(\tilde{n}_1(N), \tilde{n}_2(N)) = (\lfloor N\eta \rfloor + n_1, \lfloor N\eta \rfloor + n_2)$ and let $\alpha = \frac{\eta}{\tau}$. Note that, we do not scale the horizontal positions which avoids hard asymptotics involving the orthogonal polynomials $Q_i, \hat{Q}_i$ or the spectral measures $\mathfrak{w}, \hat{\mathfrak{w}}$. Then, we have the following theorem whose proof, based on a simple steepest descent analysis, can be found in subsection 5.10.2:

Theorem 5.4.

$$\begin{aligned}\lim_{N\to\infty} \mathcal{K}^{t(N)} \left(\left(\left(\tilde{n}_1(N), \tilde{n}_2(N)\right), i\right), \left(\left(\tilde{m}_1(N), \tilde{m}_2(N)\right), j\right)\right) &= \mathfrak{K}_\alpha \left(\left(\left(n_1, n_2\right), i\right), \left(\left(m_1, m_2\right), j\right)\right) \\ &= \int_{I^-}^{I^+} \left[-\mathbf{1}(x \geq \alpha) + \mathbf{1}\left((n_1, n_2) \geq (m_1, m_2)\right)\right] \bar{\mathcal{P}}_i(x) x^{n_2 - m_2} \tilde{\mathcal{P}}_j(x) d\mathfrak{m}(x).\end{aligned}$$

Now, to a weight $\mathcal{W}(dx)$ on (some subset of) $\mathbb{R}$ for which the moment problem is determinate and a point $r \in \mathbb{R}$ one can associate a discrete determinantal point process with kernel denoted by $\mathsf{K}_r^{\mathcal{W}}(i,j)$ (see Remark 5.71 and [30] for the exact details). Then, as explained in subsection 5.10.2, if restricted to single levels $\mathfrak{K}_\alpha\left(\left((n, n+1), i\right), \left((n, n+1), j\right)\right)$ gives rise to the determinantal ensemble with kernel $\mathsf{K}_\alpha^{\mathfrak{w}}(i,j)$ and also $\mathfrak{K}_\alpha\left(\left((n,n), i\right), \left((n,n), j\right)\right)$ gives rise to the ensemble governed by the kernel $\mathsf{K}_\alpha^{\hat{\mathfrak{w}}}(i,j)$. Thus, $\mathfrak{K}_\alpha\left(\left((n_1, n_2), i\right), \left((m_1, m_2), j\right)\right)$ provides a novel multilevel determinantal extension of these discrete ensembles, so that particles on consecutive levels interlace (by construction). Moreover, in this generality, it is the first time that these ensembles appear in a concrete interacting particle system.

5.1.5 Further results

En route, to our computation of the correlation kernel $\mathcal{K}^t$ we introduce in Section 5.8 a large class of coherent probability measures, with respect to the Markov kernels corresponding to the alternating construction. These depend on a set of parameters $(t; \alpha_1, \cdots, \alpha_{\mathfrak{N}})$ with $\mathfrak{N} \in \mathbb{N}$ such that $t \geq 0$ and Const $\geq \alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, where Const has a natural interpretation in terms of the interval of orthogonality $\mathfrak{I}$ (see Remark 5.63). Their description is through the spectral theory explained above and the single variable function:

$$\psi(x) = \psi_{t,\vec{\alpha}}(x) = \prod_{i=1}^{\mathfrak{N}} (1 - \alpha_i x) e^{-tx}. \tag{5.7}$$

By Kolmogorov's Theorem these measures give rise to a stochastic point process in $\mathbb{GT}_{\mathbf{s}}(\infty)$ denoted by Ξ^{ψ}, which specializes to Ξ^{t} when all the $\vec{\alpha}$ coordinates are identically zero. In Theorem 5.69 we show that Ξ^{ψ} is a determinantal point process with an explicit kernel $\mathcal{K}^{\psi}$.

Moreover, combining the results of Proposition 5.52 and Section 5.11.3 in the Appendix, we obtain that under a positive definiteness assumption (see Remark 5.75) for the corresponding Karlin-McGregor polynomials these sequences of measures are actually extremal.

Finally, we observe that an inhomogeneous, with position dependent jumps, two species analogue of PushASEP (with at most two particles per site) arises if one looks at the rightmost particles in the interlacing array above. In particular the evolution of the particles $\left(X_1^{(0,1)}(t), X_1^{(1,1)}(t), X_2^{(1,2)}(t), \cdots ; t \geq 0\right)$ is autonomous. Of course, the distribution of this $(1+1)$-dimensional model is completely characterized by Theorem 5.3.

5.1.6 Outlook and questions

Many directions and questions arise from the work in this chapter. We indicate and briefly discuss the ones we find the most interesting:

- (**Scaling limits**) Study different scaling regimes for the inhomogeneous process introduced above. It is clear from simulations, performed by the author, that interesting behaviour arises when one introduces for example slow or fast regions, periodic or trigonometric rates. The analysis of course boils down to the associated one-dimensional orthogonal polynomials. Another question is whether in any of the possible scaling regimes perturbations of the rates still give the same asymptotic behaviour. This again will come down to universality statements for orthogonal polynomials.

- (**Inhomogeneous TASEP**) Borodin-Ferrari studied in [21] a (2+1)-dimensional growth model taking values in a Gelfand-Tsetlin pattern. Each particle has an independent exponential clock of rate one for jumping to the right (by one) and particles as before interact through the push-block dynamics. The projection to the left most particles gives the Totally Asymmetric Simple Exclusion Process (TASEP). A natural question is to find the right, namely integrable, inhomogeneous generalization of the model in [21]. This could provide a route to some exact solvability in inhomogeneous TASEP which has thus far resisted many efforts. A particular case is the slow bond problem for which a breakthrough was achieved for the leading order behaviour using non-exactly solvable techniques, see [15].

- (**Boundary of generalized type BC-graph**) As mentioned above one can associate a branching graph to the alternating construction, that we call generalized type-BC branching graph, its multiplicities are given by general product form weights. Is it possible, at least for certain multiplicities, to describe its boundary? Moreover, what is the relation of such extreme coherent measures with dynamics on the graph. In the case of both the Gelfand-Tsetlin graph and the type-BC graph there is an exact correspondence with continuous time birth and death chain dynamics, discrete time Bernoulli and also geometric jumps. A more ambitious direction would be to develop some kind of perturbation theory for these graphs.

5.1.7 Contents of the chapter

We quickly describe the contents of each section. In Section 5.2, we introduce all the relevant material on birth and death (or bilateral) chains that we need. We then introduce the coalescing flows and give our two-level couplings formulae. We moreover obtain our intertwining and Markov functions results. In Section 5.3, we prove that the formulae describe the push-block dynamics by showing that they solve the corresponding backwards equations and that these are unique. Furthermore, we spell out a procedure for concatenating such two-level processes in order to build an interlacing array in a consistent manner. In Section 5.4, we define and collect some facts about branching graphs along with two classical examples, the Gelfand-Tsetlin graph and the BC-type branching graph, and the graph corresponding to the alternating construction. We also state the theorem of Borodin and Olshanski known as the method of intertwiners. In Section 5.5, we show how known and new examples of consistent dynamics can be obtained as corollaries of our first main result, including the ones in [28] and [50] and moreover, we characterize the ones arising from the coupling studied here that are coherent for the Gelfand-Tsetlin graph. In Sections 5.6 and 5.7, we introduce the Karlin-McGregor polynomials associated to $\mathcal{D}$ and $\hat{\mathcal{D}}$-chains and their multivariate analogues and prove some of their properties. In Section 5.8, we introduce coherent measures (with respect to the Markov kernels associated to the alternating construction) $\mathcal{M}^{\psi}_{n-1,n}, \mathcal{M}^{\psi}_{n,n}$ indexed by a function ψ and investigate some of their properties. For $\psi_t(x) = e^{-tx}$ these correspond to the distribution at time t of the push-block dynamics started from the fully packed initial condition as described in the paragraphs above. In Section 5.9, we introduce "evolution operators" for coherent measures denoted by $\mathfrak{P}^{g}_{n-1,n}, \mathfrak{P}^{g}_{n,n}$, which when applied to $\mathcal{M}^{\psi}_{n-1,n}, \mathcal{M}^{\psi}_{n,n}$ "evolve" these measures to $\mathcal{M}^{g\psi}_{n-1,n}, \mathcal{M}^{g\psi}_{n,n}$. We also obtain some sufficient conditions for functions ψ to give rise to bona fide probability measures (with positivity being the non-trivial issue here). In Section 5.10, we finally prove our second main result, the explicit computation of the correlation kernel of the process described previously, this being an application of the Eynard-Mehta theorem (see e.g. [35]) along with some preliminaries. Finally, in the Appendix we collect a couple of technical proofs along with; essentially reproducing for our own and the reader's convenience, an argument of Okounkov and Olshanski that we found in [117], that uses de Finetti's theorem to give a sufficient condition for coherent measures with multiplicative "generating functions" to be extremal, based on a kind of positive definiteness property (an assumption) for the associated orthogonal polynomials.

5.2 Coalescing birth and death chains and intertwinings

5.2.1 General facts on birth and death chains and their duals

We consider a birth and death chain on $I = \mathbb{N}$, or bilateral birth and death chain on $I = \mathbb{Z}$, denoted by X, given by the infinitesimal *birth* $(\lambda(x))_{x\in I}$ and *death* $(\mu(x))_{x\in I}$ rates and with matrix of transition rates denoted by $\mathcal{D}$,

$$\mathcal{D}(x,y) = \begin{cases} \lambda(x) & y = x+1 \\ -\lambda(x) - \mu(x) & y = x \\ \mu(x) & y = x-1 \end{cases} .$$

We assume that $\lambda(x), \mu(x) > 0$, for all $x \in \mathbb{Z}$ in the bilateral case and $\mu(0) = 0$ in case of $I = \mathbb{N}$, i.e. that 0 is reflecting ($\lambda(x)$ for $x \geq 0$ and $\mu(x) > 0$ for $x \geq 1$). We moreover assume that, ∞ is a *natural* boundary point, namely a process can neither reach in finite time or be started from such a point (similarly $-\infty$ is assumed *natural* in case $I = \mathbb{Z}$), so that the rates uniquely determine our chain. Sufficient conditions for this, will be given later on below in this subsection. In order to be more concise, we will frequently refer to such a Markov chain with generator $\mathcal{D}$, as a $\mathcal{D}$-chain. Now, define the forward and backward discrete derivatives by,

$$(\nabla f)(x) = f(x+1) - f(x), (\bar{\nabla} f)(x) = f(x-1) - f(x),\ x \in I,$$

and observe that $\mathcal{D}$ can be regarded as a difference operator acting on functions, $f : I \to \mathbb{C}$ as follows,

$$(\mathcal{D}f)(x) = \lambda(x)(\nabla f)(x) + \mu(x)(\bar{\nabla} f)(x),\ x \in I.$$

Denote by $p_t(x,y)$ the transition density of the $\mathcal{D}$-chain i.e. with $(X(t); t \geq 0)$ denoting a realization of this chain governed by the family of measures indexed by starting positions, $\{\mathbb{P}_x\}_{x\in I}$ then, $p_t(x,y) = \mathbb{P}_x(X(t) = y)$. Furthermore, we denote by $(P_t; t \geq 0)$ the Feller semi-group (that maps the space of functions vanishing at infinity to itself), it gives rise to (the fact that all these are well defined is discussed next). In particular, we will often use the notation:

$$P_t \mathbf{1}_{[l,y]}(x) = \sum_{l \leq z \leq y} p_t(x,z).$$

We note that, under the conditions (5.8) and (5.9) below, $p_t(x,y)$ will be the unique solution to the Kolmogorov backward differential equation given by, $\forall t > 0, x, y \in I$,

$$\frac{d}{dt} p_t(x,y) = \mathcal{D}_x p_t(x,y),$$

subject further to the initial condition, positivity and sub-stochasticity assumptions:

$$p_0(x,y) = \delta_{x,y} \, , \, p_t(x,y) \geq 0 \text{ and } \sum_{y \in I} p_t(x,y) \leq 1.$$

Here, $\mathcal{D}_x$ acts as $\mathcal{D}$ on a (possibly multivariate) function in the variable labelled x. Now, define the *symmetrizing* measure of the $\mathcal{D}$-chain (the measure with respect to which it is reversible) which we denote by π as follows,

$$\pi(x) = \prod_{i=1}^{x} \frac{\lambda(i-1)}{\mu(i)} \quad x \geq 1, \pi(0) = 1, \pi(x) = \prod_{i=1}^{-x} \frac{\mu(x+i)}{\lambda(x+i-1)} \, , \; x \leq -1.$$

In the case of $I = \mathbb{N}$, we will enforce throughout this chapter, the following two conditions (see [97], [156]),

$$\sum_{j=0}^{\infty} \frac{1}{\lambda(j)\pi(j)} \sum_{i=0}^{j} \pi(i) = \infty, \tag{5.8}$$

$$\sum_{j=0}^{\infty} \frac{1}{\lambda(j)\pi(j)} \sum_{i=j+1}^{\infty} \pi(i) = \infty. \tag{5.9}$$

Then, under conditions (5.8) and (5.9) the chain with generator $\mathcal{D}$ is uniquely determined by its rates, it is non-explosive and $p_t(x,y)$ is the unique (stochastic) solution to both the backwards and forwards equations (for proofs of these statements see for example [97] or [156] and the references therein). Moreover, we have $p_t(x,y) \to 0$ as $y \to \infty$ and $p_t(x,y) \to 0$ as $x \to \infty$.

In the case of a bilateral chain, in order for both $-\infty$ and $+\infty$ to be natural boundaries, which in particular, ensures the uniqueness of solutions to both the backwards and forwards equation and non-explosiveness, we need the following four conditions. The first two, (5.10) and (5.11), govern the behaviour at $+\infty$ and the last two, (5.12) and (5.13), at $-\infty$, for a proof see Theorem 2.5 and the discussion at the end of page 511 of [129],

$$\sum_{j=1}^{\infty} \frac{1}{\lambda(j)\pi(j)} \sum_{i=1}^{j} \pi(i) = \infty, \tag{5.10}$$

$$\sum_{j=1}^{\infty} \pi(j) \sum_{i=1}^{j-1} \frac{1}{\lambda(i)\pi(i)} = \infty, \tag{5.11}$$

$$\sum_{j=-\infty}^{-1} \frac{1}{\lambda(j)\pi(j)} \sum_{i=j+1}^{-1} \pi(i) = \infty, \tag{5.12}$$

$$\sum_{j=-\infty}^{-1} \pi(j) \sum_{i=j}^{-1} \frac{1}{\lambda(i)\pi(i)} = \infty. \tag{5.13}$$

We now come to the definition (going back to the papers of Karlin and McGregor [92], [93]) of the dual chain $\hat{X}$, on $\mathbb{N}^- = \mathbb{N} \cup \{-1\}$ and $\mathbb{Z}$ respectively, that is given by the infinitesimal rates $\hat{\lambda}(x) = \mu(x+1)$ and $\hat{\mu}(x) = \lambda(x)$ and with generator:

$$\hat{\mathcal{D}}(x,y) = \begin{cases} \hat{\lambda}(x) = \mu(x+1) & y = x+1 \\ -\mu(x+1) - \lambda(x) & y = x \\ \hat{\mu}(x) = \lambda(x) & y = x-1 \end{cases}.$$

Note that, in the case of $\mathbb{N}^-$ then, -1 is an *absorbing state*. As before, in order to be concise and to emphasise the role of duality in this work, we will sometimes refer to this Markov chain as the $\hat{\mathcal{D}}$-chain and denote its transition density by $\hat{p}_t(x,y)$ (in case of a birth and death chain we only consider the transition density in $\mathbb{N}$, i.e it is the same as that of the process *killed* at -1), its semigroup by $\left(\hat{P}_t; t \geq 0\right)$ and symmetrizing measure by $\hat{\pi}$.

Now, it is not hard to check that, conditions (5.8), (5.9) and (5.10),(5.11),(5.12),(5.13) respectively hold for the rates (λ, μ), if and only if they hold for the dual rates $\left(\hat{\lambda}, \hat{\mu}\right)$ and thus the dual chain is well posed with natural boundaries at $\pm\infty$ as well.

With the above definitions in place, we arrive at the following key duality relation for birth and death chains, going back to Karlin's and McGregor's classic works [92] and [93] (see also [144], [49]). The relation is also true for bilateral chains and we present, the admittedly almost identical, proof in the Appendix because we could not locate it in the literature. We also give a "graphical" proof in the next subsection.

Lemma 5.5 (Siegmund duality). *For $x, y \in I$ and $t \geq 0$ we have,*

$$P_t \mathbf{1}_{[l,y]}(x) = \hat{P}_t \mathbf{1}_{[x,\infty)}(y), \tag{5.14}$$

where $l = 0$ if $I = \mathbb{N}$ or $l = -\infty$ if $I = \mathbb{Z}$ respectively.

Remark 5.6. *Note that, the $\hat{}$ operation is not an involution even in the case of $I = \mathbb{Z}$, unlike the diffusion process setting, see Chapter 1. This is an artefact of the discrete world and will complicate things a little bit, since these asymmetries make keeping track of the positions of $\leq$ and $<$ below important.*

5.2.2 Discrete coalescing flow and two-level process

First, we define the interlacing spaces our processes will take values in, with I being either $\mathbb{N}$ or $\mathbb{Z}$, in particular all coordinates are integers, and with $l = 0$ or $-\infty$ respectively, as follows,

$$W^n(I) = \{x = (x_1, \cdots, x_n) \in I^n : l \leq x_1 < \cdots < x_n < \infty\},$$
$$W^{n,n+1}(I) = \{(x,y) = (x_1, \cdots, x_{n+1}, y_1, \cdots, y_n) \in I^{2n+1} : l \leq x_1 \leq y_1 < x_2 \leq \cdots < x_{n+1} < \infty\},$$
$$W^{n,n}(I) = \{(x,y) = (x_1, \cdots, x_n, y_1, \cdots, y_n) \in I^{2n} : l \leq y_1 \leq x_1 < y_2 \leq \cdots \leq x_n < \infty\}.$$

Also, define for $x \in W^n(I)$,

$$W^{\bullet,n}(x) = \{y \in W^{\bullet}(I) : (x, y) \in W^{\bullet,n}(I)\}.$$

Similarly, define $W^{n,\bullet}(y)$,

$$W^{n,\bullet}(y) = \{x \in W^{\bullet}(I) : (x, y) \in W^{n,\bullet}(I)\}.$$

Graphical construction of coalescing flow We now describe the "graphical" construction of the coalescing flow of birth and death (or bilateral) chains. For each site of the lattice $x \in I$, we have independent Poisson processes, indexed by time $t \in \mathbb{R}$, of up $\uparrow$ arrows denoted by $\{N_x^{\uparrow}(t) : t \in \mathbb{R}\}$ of (constant) rate $\lambda(x)$ and down $\downarrow$ arrows denoted by $\{N_x^{\downarrow}(t) : t \in \mathbb{R}\}$ of (constant) rate $\mu(x)$.

We now define the family of random maps $\{\mathbf{\Phi}_{s,t} : I \to I; s \leq t\}$ as follows. For $x \in I$ and $s \leq t$, the value $\mathbf{\Phi}_{s,t}(x)$ is arrived at by starting at time s at site x and following the direction of the arrows until time t. The site you are on at time t is defined to be $\mathbf{\Phi}_{s,t}(x)$. There is a slight ambiguity in this definition at arrival times of the arrows and by convention we take the right continuous (in time) version of this map. See Figure 5.2 for an illustration.

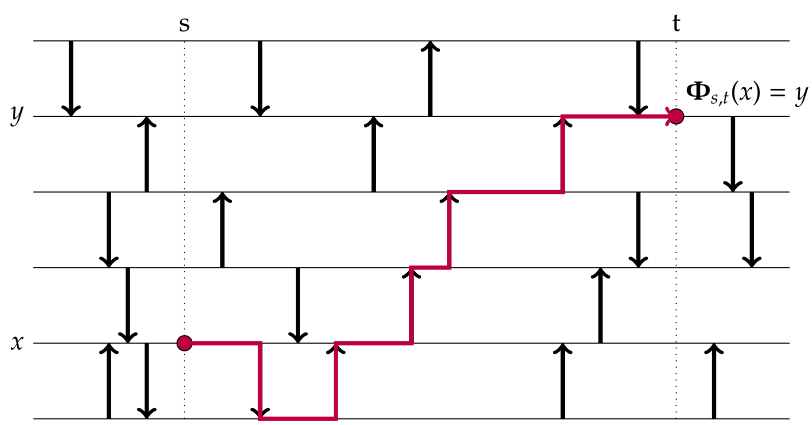


Figure 5.2: The graphical construction of the coalescing flow $(\mathbf{\Phi}_{s,t}(\cdot); s \leq t)$.

It is clear from the construction, namely from the properties of the independent Poisson processes $\{N_x^{\uparrow}, N_x^{\downarrow} : x \in I\}$, that almost surely $\mathbf{\Phi}_{\cdot,\cdot}(\cdot)$ satisfies: $\forall u \leq s \leq t \in \mathbb{R}$ and $h \in \mathbb{R}$, $\mathbf{\Phi}_{t,t} = Id$, $\mathbf{\Phi}_{s,t} \circ \mathbf{\Phi}_{u,s} = \mathbf{\Phi}_{u,t}$, $\mathbf{\Phi}_{s,t} \stackrel{\text{law}}{=} \mathbf{\Phi}_{s+h,t+h}$ and $\mathbf{\Phi}_{s,t}$ and $\mathbf{\Phi}_{u,s}$ are independent. Moreover, $\mathbf{\Phi}_{s,t}(x)$ is distributed as a $\mathcal{D}$-chain ran from time s to time t starting from x and the joint distribution of $((\mathbf{\Phi}_{s,t}(x_1), \mathbf{\Phi}_{s,t}(x_2)); t \geq s)$ is that of two independent $\mathcal{D}$-chains starting from sites x_1 and x_2 at time s, that coalesce when they meet, since once they are at the same site they will follow the same arrows.

Now, define the dual flow for $s \leq t$ by:

$$\mathbf{\Phi}^*_{s,t}(x) = \mathbf{\Phi}^{-1}_{-t,-s}(x) = \sup\{w \in I : \mathbf{\Phi}_{-t,-s}(w) \leq x\}.$$

Note that,

$$\mathbf{\Phi}^*_{s,t}\left(\mathbf{\Phi}^*_{u,s}(x)\right) = \sup\{w \in I : \mathbf{\Phi}_{-t,-s}(w) \le \mathbf{\Phi}^*_{u,s}(x)\} = \sup\{w \in I : \mathbf{\Phi}_{-s,-u} \circ \mathbf{\Phi}_{-t,-s}(w) \le x\} = \mathbf{\Phi}^*_{u,t}(x).$$

More generally, the fact that this again satisfies the stochastic flow properties will be implied immediately from the pathwise construction below, which also identifies the dynamics of the random maps $\{\mathbf{\Phi}^*_{s,t}; s \le t\}$.

The following statements are purely deterministic. Suppose that on each site of the lattice $x \in I$ we have a countable number, with no accumulation points, of up $\uparrow$ and down $\downarrow$ arrows arriving at (distinct) time points $\{\cdots < t^{x,\uparrow}_{-1} < t^{x,\uparrow}_{0} < t^{x,\uparrow}_{1} < t^{x,\uparrow}_{2} < \cdots\}$ and $\{\cdots < t^{x,\downarrow}_{-1} < t^{x,\downarrow}_{0} < t^{x,\downarrow}_{1} < t^{x,\downarrow}_{2} < \cdots\}$ respectively (by convention, $t^{x,\cdot}_0$ denotes the first arrival after time-0). Define the maps $F_{\cdot,\cdot}(\cdot)$ as before: Start at time s at site x and follow the direction of the arrows until time t. The site you are at is defined to be $F_{s,t}(x)$. As before, there is some ambiguity in this definition at the arrival times $t^{x,\uparrow}_{\cdot}, t^{x,\downarrow}_{\cdot}$ of arrows and again by convention we take the right continuous (in time) version of this map. In particular, if $t^{x,\uparrow}_l$ is the first arrow after time s at site x then $F_{s,t}(x) = x$ for $s \le t < t^{x,\uparrow}_l$ while $F_{s,t^{x,\uparrow}_l}(x) = x+1$ and so on.

Consider $F^{-1}_{s,t}(x) = \sup\{w \in I : F_{s,t}(w) \le x\}$ and our aim is to obtain a pathwise description for this map. We introduce the following two operations on the original/black arrows to get new/*red* arrows. It is important to note the minor asymmetry (coming from our choice of $\le$ in the definition of $F^{-1}_{s,t}$) in the operations below.

1. An up arrow $\uparrow$ at time t from site x to site $x+1$, becomes a *red* down arrow $\downarrow$ from site x to site $x-1$ at time t. See Figure 5.3 for an illustration.

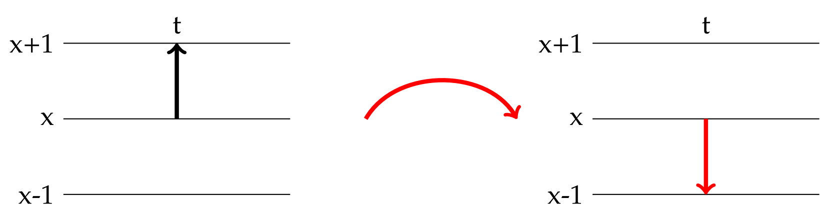


Figure 5.3: The transformation of up arrows.

2. A down arrow $\downarrow$ at time t from site $x+1$ to site x, becomes a *red* up arrow $\uparrow$ from site x to site $x+1$ at time t. See Figure 5.4 for an illustration.

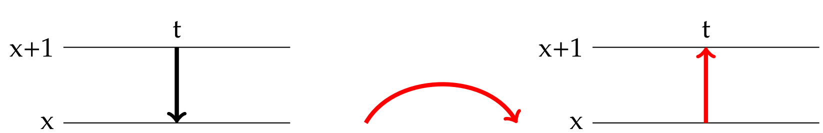


Figure 5.4: The transformation of down arrows.

Moreover, define the maps $G_{\cdot,\cdot}(\cdot)$, when evaluated at $G_{s,t}(x)$ as follows: Start at time t at site x and follow the direction of the *red* up and down arrows backwards until time s. The site you are at, is defined to be $G_{s,t}(x)$.

We then have the following proposition, whose proof is deferred to the Appendix.

Proposition 5.7. *For $x \in I$ and $s \le t$, we have $F_{s,t}^{-1}(x) = G_{s,t}(x)$.*

Observe that, if the processes $N_x^{\uparrow}$ of up arrows are independent Poisson processes of rate $\lambda(x)$ and down arrows $N_x^{\downarrow}$ are of rate $\mu(x)$ then the processes of *red* arrows $N_x^{\uparrow}, N_x^{\downarrow}$, that are followed by $\mathbf{\Phi}^*$, are independent Poisson processes with rates $\mu(x+1)$ and $\lambda(x)$ respectively. Thus, this construction identifies the dual flow as that of coalescing $\hat{\mathcal{D}}$-chains ran backwards in time. In particular, this also gives a graphical proof of the Siegmund duality Lemma 5.5.

Remark 5.8. *It is possible, and equivalent, to consider the dual flow $\mathbf{\Phi}^*$ on the (dual) lattice $I \pm \frac{1}{2}$. Then, the operations performed to obtain arrows followed by this flow backwards in time become symmetric.*

We arrive at the following proposition for the finite dimensional distributions of the coalescing flow. The result is stated for times 0 and t, but by stationarity it extends to arbitrary pairs of times.

Proposition 5.9. *For $z, z' \in W^n(I)$,*

$$\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(z_i) \le z_i' \text{ , for } 1 \le i \le n\Big) = \det\left(P_t \mathbf{1}_{[l,z_j']}(z_i) - \mathbf{1}(i<j)\right)_{i,j=1}^n.$$

Proof. By translating the non-intersection probability found in display (3) in [94] and the paragraph following it, to our setting we get for $(y_1, \cdots, y_n) \in W^n(I)$:

$$\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(z_i) = y_i, \text{ for } 1 \le i \le n\Big) = \det\left(p_t(z_i, y_j)\right)_{i,j=1}^n .$$

This is because of the following observation: the fact that the $\mathbf{\Phi}_{0,t}(z_i)$ are equal to distinct points y_i is equivalent to non-coalescence/non-intersection in the time interval $[0,t]$ of the underlying independent $\mathcal{D}$-chains. Then, summing over $(y_1, \cdots, y_n)$ in $\{l \le y_1 \le z_1', z_1'+1 \le y_2 \le z_2', \cdots, z_{n-1}'+1 \le y_n \le z_n'\}$ and successively adding column j to column $j+1$ we obtain,

$$\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(z_1) \le z_1' < \mathbf{\Phi}_{0,t}(z_2) \le z_2' < \cdots < \mathbf{\Phi}_{0,t}(z_n) \le z_n'\Big) = \det\left(P_t \mathbf{1}_{[l,z_j']}(z_i)\right)_{i,j}^n.$$

The result will then follow, by writing the indicator function of the event,

$$\{\mathbf{\Phi}_{0,t}(z_1) \le z_1', \mathbf{\Phi}_{0,t}(z_2) \le z_2', \cdots, \mathbf{\Phi}_{0,t}(z_n) \le z_n'\},$$

in terms of an expansion of indicator functions of events of the form,

$$\Big\{\mathbf{\Phi}_{0,t}(z_{i_1}) \le z_{j_1}' < \mathbf{\Phi}_{0,t}(z_{i_2}) \le z_{j_2}' < \cdots < \mathbf{\Phi}_{0,t}(z_{i_k}) \le z_{i_k}'\Big\},$$

for increasing subsequences $i_1, \cdots, i_k$ and $j_1, \cdots, j_k$. This combinatorial fact is presented in detail in Proposition 9 of [164], to which the reader is referred to. ◻

We now come to the key definition of the time-dependent block determinant kernel, $\mathsf{q}_t^{n,n+1}((x,y),(x',y'))$ on $W^{n,n+1}(I)$.

Definition 5.10. *For $(x,y),(x',y')\in W^{n,n+1}(I)$ and $t\geq 0$, define $\mathsf{q}_t^{n,n+1}((x,y),(x',y'))$ by,*

$$\begin{aligned}&\mathsf{q}_t^{n,n+1}((x,y),(x',y'))=\\&=\frac{\prod_{i=1}^n\hat{\pi}(y_i')}{\prod_{i=1}^n\hat{\pi}(y_i)}(-1)^n\nabla_{y_1}\cdots\nabla_{y_n}(-1)^{n+1}\bar{\nabla}_{x_1'}\cdots\bar{\nabla}_{x_{n+1}'}\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(x_i)\leq x_i',\mathbf{\Phi}_{0,t}(y_j)\leq y_j' \text{ for all } i,j\Big).\end{aligned}$$

Note that,

$$\mathsf{q}_t^{n,n+1}((x,y),(x',y'))=\frac{\prod_{i=1}^n\hat{\pi}(y_i')}{\prod_{i=1}^n\hat{\pi}(y_i)}\mathbb{P}\Big(\mathbf{\Phi}_{0,t}(x_i)=x_i',\mathbf{\Phi}^*_{-t,0}(y_j')=y_j \text{ for all } i,j\Big) \tag{5.15}$$

and that, using Proposition 5.9, $\mathsf{q}_t^{n,n+1}$ can be written out explicitly,

$$\mathsf{q}_t^{n,n+1}((x,y),(x',y'))=\det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(y,x') & \mathsf{D}_t(y,y')\end{pmatrix}, \tag{5.16}$$

where, using reversibility with respect to $\hat{\pi}$,

$$\begin{aligned}\mathsf{A}_t(x,x')_{ij}&=p_t(x_i,x_j')=-\bar{\nabla}_{x_j'}P_t\mathbf{1}_{[l,x_j']}(x_i),\\ \mathsf{B}_t(x,y')_{ij}&=\hat{\pi}(y_j')(P_t\mathbf{1}_{[l,y_j']}(x_i)-\mathbf{1}(j\geq i)),\\ \mathsf{C}_t(y,x')_{ij}&=\hat{\pi}^{-1}(y_i)\nabla_{y_i}\bar{\nabla}_{x_j'}P_t\mathbf{1}_{[l,x_j']}(y_i),\\ \mathsf{D}_t(y,y')_{ij}&=-\frac{\hat{\pi}(y_j')}{\hat{\pi}(y_i)}\nabla_{y_i}P_t\mathbf{1}_{[l,y_j']}(y_i)=\hat{p}_t(y_i,y_j').\end{aligned}$$

We define the family of operators $\Big(\mathsf{Q}_t^{n,n+1};t\geq 0\Big)$, acting on bounded Borel functions on $W^{n,n+1}(I)$ by,

$$(\mathsf{Q}_t^{n,n+1}f)(x,y)=\sum_{(x',y')\in W^{n,n+1}(I)}\mathsf{q}_t^{n,n+1}((x,y),(x',y'))f(x',y').$$

We will say that the family of operators $(\mathfrak{P}(t);t\geq 0)$ defined on bounded Borel functions on a space $\mathcal{X}$ forms a sub-Markov semigroup on $\mathcal{X}$ if the following hold:

$$\begin{aligned}&\mathfrak{P}(0)=Id,\\&\mathfrak{P}(t)1\leq 1\ ,\ \text{for } t\geq 0,\\&\mathfrak{P}(t)f\geq 0\ ,\ \text{for} f\geq 0,\\&\mathfrak{P}(t+s)=\mathfrak{P}(t)\mathfrak{P}(s),\ \text{for } s,t\geq 0.\end{aligned} \tag{5.17}$$

Proposition 5.11. $\Big(\mathsf{Q}_t^{n,n+1};t\geq 0\Big)$ *forms a sub-Markov semigroup on $W^{n,n+1}(I)$. We can thus associate to it a Markov process $(X,Y)=((X(t),Y(t))\,;t\geq 0)$, with possibly finite lifetime, with state* *space* $W^{n,n+1}(I)$.

Proof. We proceed to check the items found in display (5.17). The initial, or *time-0*, condition follows immediately from the representation (5.15). The second property, follows from performing the sum $\sum_{x'\in W^{\star,n}(y')}$ and then we are left with the sum,

$$\sum_{y'\in W^n(I)} \det(\hat{p}_t(y_i,y'_j))^n_{i,j} \le 1, \forall y \in W^n, t \ge 0.$$

The quite non-trivial at first sight *positivity* preserving property again follows from representation (5.15). The semigroup property for the transition kernels $\mathsf{q}_t^{n,n+1}$, can be got in the following fashion. First, by making use of the composition identity $\mathbf{\Phi}_{0,s+t} = \mathbf{\Phi}_{s,s+t} \circ \mathbf{\Phi}_{0,s}$, then using the independence of $\mathbf{\Phi}_{s,s+t}$ and $\mathbf{\Phi}_{0,s}$, noting that $\mathbf{\Phi}_{s,s+t} \overset{\text{law}}{=} \mathbf{\Phi}_{0,t}$ and conditioning on the values of $\mathbf{\Phi}_{0,s}(x_i)$ and $\mathbf{\Phi}^*_{-(s+t),-s}(y''_j)$ we obtain,

$$\begin{aligned}
&\mathsf{q}_{s+t}^{n,n+1}((x,y),(x'',y'')) = \frac{\prod_{i=1}^n \hat{\pi}(y''_i)}{\prod_{i=1}^n \hat{\pi}(y_i)}\mathbb{P}\Big(\mathbf{\Phi}_{0,s+t}(x_i) = x''_i, \mathbf{\Phi}^*_{-(s+t),0}(y''_j) = y_j \text{ for all } i,j\Big)\\
&= \frac{\prod_{i=1}^n \hat{\pi}(y''_i)}{\prod_{i=1}^n \hat{\pi}(y_i)} \sum_{(x',y')\in W^{n,n+1}(I)} \mathbb{P}\Big(\mathbf{\Phi}_{0,s}(x_i) = x'_i, \mathbf{\Phi}_{s,s+t}(x'_i) = x''_i, \mathbf{\Phi}^*_{-s,0}(y'_j) = y_j, \mathbf{\Phi}^*_{-(s+t),-s}(y''_j) = y'_j\Big)\\
&= \sum_{(x',y')\in W^{n,n+1}(I)} \frac{\prod_{i=1}^n \hat{\pi}(y'_i)}{\prod_{i=1}^n \hat{\pi}(y_i)}\mathbb{P}\Big(\mathbf{\Phi}_{0,s}(x_i) = x'_i, \mathbf{\Phi}^*_{-s,0}(y'_j) = y_j\Big)\\
&\times \frac{\prod_{i=1}^n \hat{\pi}(y''_i)}{\prod_{i=1}^n \hat{\pi}(y'_i)}\mathbb{P}\Big(\mathbf{\Phi}_{s,s+t}(x'_i) = x''_i, \mathbf{\Phi}^*_{-(s+t),-s}(y''_j) = y'_j\Big)\\
&= \sum_{(x',y')\in W^{n,n+1}(I)} \mathsf{q}_s^{n,n+1}((x,y),(x',y'))\mathsf{q}_t^{n,n+1}((x',y'),(x'',y'')).
\end{aligned}$$

The reason we are restricting our sum, in the second line onwards, over $(x',y') \in W^{n,n+1}(I)$ is because by the coalescing property for $(x,y) \in W^{n,n+1}(I)$ we have that almost surely $\{\mathbf{\Phi}_{s,t}(x_i) = x'_i, \mathbf{\Phi}^*_{-t,-s}(y'_i) = y_i\}$ is empty unless $(x',y') \in W^{n,n+1}(I)$. This then, concludes the proof of the proposition. □

We now aim to define a family of time-dependent kernels, $\mathsf{q}_t^{n,n}((x,y),(x',y'))$ on $W^{n,n}(I)$. We again, consider in a similar fashion a (discrete) *stochastic coalescing flow* $\hat{\mathbf{\Phi}}_{s,t}$, now consisting of coalescing $\hat{\mathcal{D}}$-chains. Now, define its dual as follows (**note well** the minor but important asymmetry to the above considerations) $\hat{\mathbf{\Phi}}^*_{s,t}(y) = \inf\{w : \hat{\mathbf{\Phi}}_{-t,-s}(w) \ge y\}$. As before, we have an explicit formula for its finite dimensional distributions (also by stationarity the proposition extends to arbitrary pairs of times $s \le t$).

Proposition 5.12. *For* $z, z' \in W^n(I)$,

$$\mathbb{P}\Big(\hat{\mathbf{\Phi}}_{0,t}(z_i) \ge z'_i \text{ , for } 1 \le i \le n\Big) = \det\Big(\hat{P}_t \mathbf{1}_{[z'_j,\infty)}(z_i) - \mathbf{1}(j<i)\Big)^n_{i,j=1}.$$

Proof. The proof is entirely analogous to the proof of the Proposition 5.9 for $\mathbf{\Phi}$. □

As before, we define the following kernels:

Definition 5.13. *For $(x, y), (x', y') \in W^{n,n}(I)$ and $t \geq 0$, define $\mathsf{q}_t^{n,n}((x, y), (x', y'))$ by,*

$$\mathsf{q}_t^{n,n}((x, y), (x', y')) =$$
$$= \frac{\prod_{i=1}^n \pi(y_i')}{\prod_{i=1}^n \pi(y_i)} (-1)^n \bar{\nabla}_{y_1} \cdots \bar{\nabla}_{y_n} (-1)^n \nabla_{x_1'} \cdots \nabla_{x_n'} \mathbb{P}\Big(\hat{\mathbf{\Phi}}_{0,t}(x_i) \geq x_i', \hat{\mathbf{\Phi}}_{0,t}(y_j) \geq y_j' \text{ for all } i, j\Big).$$

Observe that,

$$\mathsf{q}_t^{n,n}((x, y), (x', y')) = \frac{\prod_{i=1}^n \pi(y_i')}{\prod_{i=1}^n \pi(y_i)} \mathbb{P}\Big(\hat{\mathbf{\Phi}}_{0,t}(x_i) = x_i', \hat{\mathbf{\Phi}}^*_{-t,0}(y_j') = y_j \text{ for all } i, j\Big). \tag{5.18}$$

and that $\mathsf{q}_t^{n,n}$ can be written out explicitly,

$$\mathsf{q}_t^{n,n}((x, y), (x', y')) = \det \begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(y, x') & \mathsf{D}_t(y, y') \end{pmatrix}, \tag{5.19}$$

where,

$$\begin{aligned}
\mathsf{A}_t(x, x')_{ij} &= \hat{p}_t(x_i, x_j') = -\nabla_{x_j'} \hat{P}_t \mathbf{1}_{[x_j',\infty)}(x_i), \\
\mathsf{B}_t(x, y')_{ij} &= \pi(y_j')(\hat{P}_t \mathbf{1}_{[y_j',\infty)}(x_i) - \mathbf{1}(j \leq i)), \\
\mathsf{C}_t(y, x')_{ij} &= \pi^{-1}(y_i) \bar{\nabla}_{y_i} \nabla_{x_j'} \hat{P}_t \mathbf{1}_{[x_j',\infty)}(y_i), \\
\mathsf{D}_t(y, y')_{ij} &= -\frac{\pi(y_j')}{\pi(y_i)} \bar{\nabla}_{y_i} \hat{P}_t \mathbf{1}_{[y_j',\infty)}(y_i) = p_t(y_i, y_j').
\end{aligned}$$

Define the family of operators $\left(\mathsf{Q}_t^{n,n}; t \geq 0\right)$, acting on bounded Borel functions on $W^{n,n}(I)$ by,

$$(\mathsf{Q}_t^{n,n} f)(x, y) = \sum_{W^{n,n}(I)} \mathsf{q}_t^{n,n}((x, y), (x', y')) f(x', y').$$

Then, with completely analogous considerations as for $\left(\mathsf{Q}_t^{n,n}; t \geq 0\right)$, we get that:

Proposition 5.14. $\left(\mathsf{Q}_t^{n,n}; t \geq 0\right)$ *forms a sub-Markov semigroup on $W^{n,n}(I)$. We can thus associate to it a Markov process $(X, Y) = ((X(t), Y(t)); t \geq 0)$, with possibly finite lifetime, with state space $W^{n,n}(I)$.*

5.2.3 Intertwinings

This subsection is essentially the exact analogue, in the discrete space setting, of Section 1.2.4 of the first chapter of this thesis.

We first denote the Karlin-McGregor semigroup associated to n $\mathcal{D}$-chains by $\left(P_t^n; t \geq 0\right)$, that is given by the following transition density, with $x, y \in W^n(I)$ and $t \geq 0$,

$$p_t^n(x, y) = \det(p_t(x_i, y_j))_{i,j=1}^n.$$

Similarly, define the Karlin-McGregor semigroup $\left(\hat{P}_t^n; t \geq 0\right)$ associated to n $\hat{\mathcal{D}}$-chains (*killed* at -1 if -1 is an absorbing boundary point) given by its transition density, with $x, y \in W^n(I)$ and $t \geq 0$,

$$\hat{p}_t^n(x, y) = \det(\hat{p}_t(x_i, y_j))_{i,j=1}^n.$$

Now, define the positive kernels $\Lambda_{n,\star}$ (not necessarily of finite mass in the case of $\Lambda_{n,n}$) acting on Borel functions on $W^{n,\star}(I)$, whenever f is summable, by where $\star \in \{n, n+1\}$,

$$(\Lambda_{n,n+1} f)(x) = \sum_{y \in W^{n,n+1}(x)} \prod_{i=1}^{n} \hat{\pi}(y_i) f(x, y), \ x \in W^{n+1}(I),$$

$$(\Lambda_{n,n} f)(x) = \sum_{y \in W^{n,n}(x)} \prod_{i=1}^{n} \pi(y_i) f(x, y), \ x \in W^{n}(I).$$

Note that $\Lambda_{n,n+1}$ involves $\hat{\pi}$ while $\Lambda_{n,n}$ involves π. Moreover, observe that we can alternatively view $\Lambda_{n,\star}$ as kernels from $W^\star$ to $W^{n,\star}$, assigning to each $\mathfrak{x} \in W^\star$ a positive measure $\Lambda_{n,\star}(\mathfrak{x}, \cdot)$ on $W^{n,\star}$ supported on $\{(x, y) \in W^{n,\star} : x = \mathfrak{x}\}$. Finally, abusing notation it is obvious that we can also view $\Lambda_{n,\star}$ as kernels from $W^\star$ to W^n or as operators acting on Borel functions on W^n.

Now, consider the projection operators $\Pi_{\star,n}$, acting on bounded Borel functions on $W^\star$, induced by the projections on the Y-level, with $\star \in \{n-1, n\}$,

$$(\Pi_{\star,n} f)(x, y) = f(y), \ (x, y) \in W^{\star,n}.$$

Proposition 5.15. *For $t \geq 0$, we have the following equalities,*

$$\Pi_{n-1,n} \hat{P}_t^{n-1} = \mathsf{Q}_t^{n-1,n} \Pi_{n-1,n}, \tag{5.20}$$

$$\Pi_{n,n} P_t^n = \mathsf{Q}_t^{n,n} \Pi_{n,n}. \tag{5.21}$$

Proof. These follow directly from the probabilistic representations (5.15) and (5.18); essentially we are taking the marginal.

Alternatively, we can take the sum $\sum_{x' \in W^{\star,n}(y')}$ in the explicit form of the transition kernels and use multilinearity of the determinant. For example, in the case of $\mathsf{Q}_t^{n-1,n}$ the statement of the proposition is a consequence of the following:

$$\sum_{x'_j=y'_{j-1}+1}^{y'_j} \mathsf{A}_t(x,x')_{ij} = P_t\mathbf{1}_{[l,y'_j]}(x_i) - P_t\mathbf{1}_{[l,y'_j]}(x_i),$$
$$\sum_{x'_j=y'_{j-1}+1}^{y'_j} \mathsf{C}_t(y,x')_{ij} = -\hat{\pi}^{-1}(y_i)\nabla_{y_i}P_t\mathbf{1}_{[l,y'_j]}(y_i) + \hat{\pi}^{-1}(y_i)\nabla_{y_i}P_t\mathbf{1}_{[l,y'_{j-1}]}(y_i).$$

The case of $\mathsf{Q}_t^{n,n}$ is analogous. □

Remark 5.16. *This, being an instance of Dynkin's criterion, has the following probabilistic interpretation. The evolution of the Y-level is Markovian with respect to the filtration generated by the process (X,Y). In the case of $W^{n-1,n}$, Y evolves as $n-1$ $\hat{\mathcal{D}}$-chains killed when they intersect or when they hit -1 if -1 is absorbing and in the case of $W^{n,n}$ it evolves as n $\mathcal{D}$-chains killed when they intersect. In particular, the finite lifetime of the joint process (X,Y) corresponds to the killing time of Y.*

Moreover, the following (intermediate) intertwining relations hold.

Proposition 5.17. *For $t \geq 0$, we have the equalities of positive kernels,*

$$P_t^{n+1}\Lambda_{n,n+1} = \Lambda_{n,n+1}\mathsf{Q}_t^{n,n+1}, \tag{5.22}$$
$$\hat{P}_t^{n}\Lambda_{n,n} = \Lambda_{n,n}\mathsf{Q}_t^{n,n}. \tag{5.23}$$

Proof. This, similarly to the Proposition above, directly follows from the probabilistic representations (5.15) and (5.18).

Otherwise, we can take the sum $\sum_{y\in W^{n,\star}(x)}$ using the explicit form of the transition densities and multilinearity. In particular, (5.22) is a consequence of:

$$\sum_{y_i=x_i}^{x_{i+1}-1} \hat{\pi}(y_i)\mathsf{C}_t(y,x')_{ij} = \nabla_{x'_j}P_t\mathbf{1}_{[l,x'_j]}(x_{i+1}) - \nabla_{x'_j}P_t\mathbf{1}_{[l,x'_j]}(x_i),$$
$$\sum_{y_i=x_i}^{x_{i+1}-1} \hat{\pi}(y_i)\mathsf{D}_t(y,y')_{ij} = -\hat{\pi}(y'_j)P_t\mathbf{1}_{[l,x'_j]}(x_{i+1}) + \hat{\pi}(y'_j)P_t\mathbf{1}_{[l,y'_j]}(x_i).$$

The proof of (5.23) is analogous. □

Combining the two preceding propositions, we straightforwardly obtain the following intertwining relations for the Karlin-McGregor semigroups (where as remarked above we simply write $\Lambda_{n,\star}$ for $\Lambda_{n,\star}\Pi_{n,\star}$), for $t \geq 0$,

$$P_t^{n+1}\Lambda_{n,n+1} = \Lambda_{n,n+1}\hat{P}_t^{n}, \tag{5.24}$$
$$\hat{P}_t^{n}\Lambda_{n,n} = \Lambda_{n,n}P_t^{n}. \tag{5.25}$$

This gives us a machine, for constructing positive eigenfunctions for these semigroups; in particular it is immediate that, with $\mathbf{1}(\cdot)$ denoting the function which is constant and equal to 1 on I,

$$h_{n,n+1}(\cdot) = (\Lambda_{n,n+1}\Lambda_{n,n}\cdots\Lambda_{1,1}\mathbf{1})(\cdot), \tag{5.26}$$

$$h_{n,n}(\cdot) = (\Lambda_{n,n}\Lambda_{n-1,n}\cdots\Lambda_{1,1}\mathbf{1})(\cdot), \tag{5.27}$$

are *positive harmonic* functions for P_t^{n+1} and $\hat{P}_t^n$ respectively. In the case of birth and death chains, these functions will come up in terms of the multivariate Karlin-McGregor polynomials, in relation to a general random growth process with a wall, in section 5.7.

Before proceeding, we need to make precise one more notion, referenced several times already. For a sub-Markovian semigroup $(\mathfrak{P}(t); t \geq 0)$, with a strictly positive eigenfunction $\mathfrak{h}$, with eigenvalue e^{ct}, we define its Doob's h-transform, $\left(\mathfrak{P}^{\mathfrak{h}}(t); t \geq 0\right)$ by,

$$\left(\mathfrak{P}^{\mathfrak{h}}(t); t \geq 0\right) \stackrel{\text{def}}{=} \left(e^{-ct}\mathfrak{h}^{-1} \circ \mathfrak{P}(t) \circ \mathfrak{h}; t \geq 0\right),$$

which now, a fact which can be readily checked, forms an honest Markov semigroup, $\mathfrak{P}^{\mathfrak{h}}(t)1 = 1$ (the definition extends to non time-dependent positive kernels).

Now, coming back to our two-level process, suppose $\hat{h}_n$ is a strictly positive eigenfunction for $\hat{P}_t^n$ namely, $\hat{P}_t^n\hat{h}_n = e^{\lambda_n t}\hat{h}_n$ then,

$$\left(P_t^{n+1}\Lambda_{n,n+1}\hat{h}_n\right)(\cdot) = e^{\lambda_n t}\left(\Lambda_{n,n+1}\hat{h}_n\right)(\cdot),$$

so that, $\Lambda_{n,n+1}\hat{h}_n$ is a strictly positive eigenfunction of P_t^{n+1}. Moreover, observe that if $\hat{h}_n$ is a positive eigenfunction for $\hat{P}_t^n$ then it is an eigenfunction (with the same eigenvalue) for $\mathsf{Q}_t^{n,n+1}$. We can thus define an honest Markov process, with semigroup $\left(\mathsf{Q}_t^{n,n+1,\hat{h}_n}; t \geq 0\right)$, which is the h-transform of $\left(\mathsf{Q}_t^{n,n+1}; t \geq 0\right)$ by $\hat{h}_n$. Also, define the strictly positive function $h_{n+1}(\cdot)$ by,

$$h_{n+1}(x) = (\Lambda_{n,n+1}\hat{h}_n)(x), \ x \in W^{n+1}(I),$$

and the Markov kernel $\Lambda_{n,n+1}^{\hat{h}_n}$ by (from the definition $h_{n+1}(x) = (\Lambda_{n,n+1}\hat{h}_n)(x)$ it is immediate that $\Lambda_{n,n+1}^{\hat{h}_n}\mathbf{1} = \mathbf{1}$),

$$(\Lambda_{n,n+1}^{\hat{h}_n}f)(x) = \frac{1}{h_{n+1}(x)}\sum_{y \in W^{n,n+1}(x)}\prod_{i=1}^{n}\hat{\pi}(y_i)\hat{h}_n(y)f(x,y), \ x \in W^{n+1}(I).$$

Finally, defining $\left(P_t^{n+1,h_{n+1}}; t \geq 0\right)$ to be the Karlin-McGregor semigroup $\left(P_t^{n+1}; t \geq 0\right)$ that is h-transformed by h_{n+1}, we arrive at our first main result.

Theorem 5.18. *Let $\hat{h}_n$ be a strictly positive eigenfunction of $\hat{P}_t^n$, then with the notations of the* *paragraph above, we have the intertwining relations, for $t \geq 0$,*

$$P_t^{n+1,h_{n+1}}\Lambda_{n,n+1}^{\hat{h}_n} = \Lambda_{n,n+1}^{\hat{h}_n}\mathsf{Q}_t^{n,n+1,\hat{h}_n}, \tag{5.28}$$

$$P_t^{n+1,h_{n+1}}\Lambda_{n,n+1}^{\hat{h}_n} = \Lambda_{n,n+1}^{\hat{h}_n}\hat{P}_t^{n,\hat{h}_n}. \tag{5.29}$$

Proof. These are immediate consequences of relations (5.22) and (5.24) respectively and the discussion above. □

Moreover, using the theorem just obtained and the Rogers and Pitman Markov functions theory (see Theorem 2 in [136] for example) we immediately get the following proposition as a corollary.

Proposition 5.19. *Consider a Markov process (X, Y) with semigroup $\left(\mathsf{Q}_t^{n,n+1,\hat{h}_n}; t \geq 0\right)$. Then, the projection on the X-components evolves as a Markov process with semigroup $\left(P_t^{n+1,h_{n+1}}; t \geq 0\right)$ started from x, if (X, Y) is initialized according to $\Lambda_{n,n+1}^{\hat{h}_n}(x,\cdot)$. Moreover, in such case, for any fixed $T \geq 0$, the conditional distribution of $(X(T), Y(T))$ given $X(T)$ is $\Lambda_{n,n+1}^{\hat{h}_n}(X(T),\cdot)$*

Proof. This is a straightforward application of Theorem 2 of [136], by virtue of the intertwining relation (5.28) above, the Markov function ϕ, being the projection on the X-component, $\phi(x, y) = x$. For the conditional distribution statement see Remark (ii) on page 575 immediately after Theorem 2 of [136]. □

Similarly, in the setting of having an equal number of particles for the two levels (i.e. for a process in $W^{n,n}(I)$); if g_n is a positive eigenfunction of P_t^n and assuming $\hat{g}_n(x) = (\Lambda_{n,n}g_n)(x)$ is finite, with the analogous definitions as above, we obtain the following theorem.

Theorem 5.20. *Let g_n be a strictly positive eigenfunction of P_t^n. Then, for $t \geq 0$,*

$$\hat{P}_t^{n,\hat{g}_n}\Lambda_{n,n}^{g_n} = \Lambda_{n,n}^{g_n}\mathsf{Q}_t^{n,n,g_n}, \tag{5.30}$$

$$\hat{P}_t^{n,\hat{g}_n}\Lambda_{n,n}^{g_n} = \Lambda_{n,n}^{g_n}P_t^{n,g_n}. \tag{5.31}$$

In particular, the projection on the X-components evolves as a Markov process with semigroup $\left(\hat{P}_t^{n,\hat{g}_n}; t \geq 0\right)$ started from x, if (X, Y) is initialized according to $\Lambda_{n,n}^{g_n}(x,\cdot)$. Furthermore, for any fixed time $T \geq 0$, the conditional distribution of $(X(T), Y(T))$ given $X(T)$ is $\Lambda_{n,n}^{g_n}(X(T),\cdot)$.

Remark 5.21. *We now explain the shortest path to a complete proof of the single level intertwining relations (5.29), (5.31), or more precisely to the proof of (5.24), (5.25). There are two essential ingredients, the Siegmund duality Lemma 5.5 and the rather ingenious introduction of the $\mathsf{q}_t^{\bullet,\star}((x, y),(x', y'))$ transition kernels. Once, we define $\mathsf{q}_t^{\bullet,\star}((x, y),(x', y'))$ by (5.15) or (5.18), none of its probabilistic properties or the coalescing flows picture are needed. We can then proceed as in the proofs of Propositions 5.15 and 5.17 by taking the sums over x' and y, **assuming these sums converge**. Of course, if $\mathsf{q}_t^{\bullet,\star}((x, y),(x', y'))$ is positive we can make use of Tonelli's theorem to interchange the sums, however with the possibility that both sides are infinite.*

We also comment on the relation to Borodin and Olshanski's approach in [28] (also Cuenca's in [50]). Their proof checks the intertwining relation at the multivariate infinitesimal level and then concludes by a lift to semigroups. Both of our proofs of the Siegmund duality Lemma 5.5 in the Appendix also contain such a lift, but in the single variable setting. The introduction of the explicit coupling, equivalently of $\mathsf{q}_t^{\bullet,\star}((x,y),(x',y'))$, *is what allows us to essentially check such a relation only in a single variable.*

Remark 5.22. *By the methods presented above, we have identified the finite lifetime of the process* $Z = (X, Y)$ *as the lifetime of the autonomous component* Y*, which we have described explicitly. Moreover, under special initial conditions we have proven that the projection on the* X*-level turns out to be a Markov process as well, but the interaction between* X *and* Y *still remains unclear. It is natural to guess, from the locality of the coalescing flow (namely that particles only interact whence they meet) and the fact that the* Y*-level is autonomous, that the* X*-particles should be blocked and pushed, in order for the interlacing to remain. This turns out to be exactly the case and we pursue it next.*

5.3 Push-block dynamics

5.3.1 Push-block dynamics for the two-level process

In this subsection, we prove that the $\mathsf{q}_t^{n,n+1}$ transition matrix governs the dynamics of a continuous time, possibly with finite lifetime, Markov chain (X, Y) in $W^{n,n+1}$ described informally as follows: The Y-level consists of n independent $\hat{\mathcal{D}}$-chains and the X-level of $n+1$ independent $\mathcal{D}$-chains that are *"pushed"* and *"blocked"* by the Y-particles, when the process is at the *boundary* (precised below) $\partial W^{n,n+1}$, in order for it to remain in $W^{n,n+1}$. The chain is *killed* when two Y-particles collide or hit $l^* = l - 1$ i.e. at the stopping time,

$$\mathfrak{T}_{W^{n,n+1}} = \inf\{t > 0 : \exists\, 1 \le i < j \le n \text{ , such that } Y_i(t) = Y_j(t) \text{ or } Y_i(t) = l^*\}.$$

See Figures 5.5-5.8 for an illustration of the four possible types (pushing and blocking from the left and from the right) of interaction between X-particles and Y-particles in $W^{n,n+1}$.

Similarly, the $\mathsf{q}_t^{n,n}$ transition matrix governs the dynamics of a continuous time, possibly with finite lifetime, Markov chain (X, Y) in $W^{n,n}$ with the following informal description: The Y-level consists of n independent $\mathcal{D}$-chains and the X-level of n independent $\hat{\mathcal{D}}$-chains that are *"pushed"* and *"blocked"* by the Y-particles, when the process is at $\partial W^{n,n}$, in order for it to remain in $W^{n,n}$. The chain is *killed* when two Y-particles collide i.e. at the stopping time (note that compared to $W^{n,n+1}$, now $Y_1(t)$ never reaches $l^* = l - 1$ since the $\mathcal{D}$-chain is reflecting at l),

$$\mathfrak{T}_{W^{n,n}} = \inf\{t > 0 : \exists\, 1 \le i < j \le n \text{ , such that } Y_i(t) = Y_j(t)\}.$$

See Figures 5.9-5.12 for an illustration of the possible interactions in $W^{n,n}$ and also note the

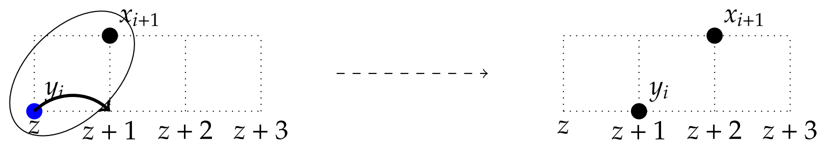


Figure 5.5: In $W^{n,n+1}$, a jump of y_i pushes (induces a simultaneous jump of) x_{i+1} to the right so that the interlacing remains. Here, the jump happens with rate $\hat{\lambda}(z) = \mu(z+1)$.

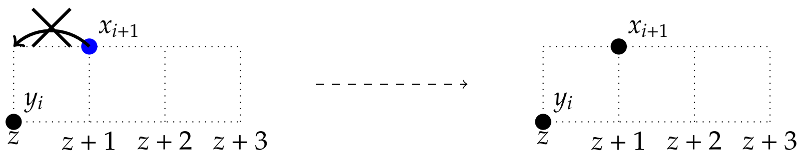


Figure 5.6: In $W^{n,n+1}$, a jump of x_{i+1} to the left is blocked by y_i so that the interlacing remains. Here, the clock of x_{i+1} rings with rate $\mu(z+1)$.

asymmetry (again related to the locations of $\leq$ and strict $<$ in the definitions of $W^{n,n}$, $W^{n,n+1}$) compared to the dynamics in $W^{n,n+1}$.

We will only consider the dynamics in $W^{n,n+1}$ in detail, as the case of $W^{n,n}$ is entirely analogous (but see Remark 5.26 below for a discussion). We define the *boundary of* $W^{n,n+1}$ denoted by $\partial W^{n,n+1}$, as follows,

$$\partial W^{n,n+1} = \{(x,y) \in W^{n,n+1} : \exists 1 \leq i \leq n+1, \text{ such that with } x'_i = x_i \pm 1 \text{ then } (x',y) \notin W^{n,n+1}\}.$$

Also, define the *interior of* $\mathring{W}^{n,n+1}$ by $\mathring{W}^{n,n+1} = W^{n,n+1} \backslash \partial W^{n,n+1}$. Finally, define the following indexing sets, $I^{n,n+1,+}_{adm}(x,y)$ and $I^{n,n+1,-}_{adm}(x,y)$ for $(x,y) \in W^{n,n+1}$ ("adm" stands for admissible jump),

$$\begin{aligned} I^{n,n+1,+}_{adm}(x,y) &= \{1 \leq i \leq n+1 : (x',y) \in W^{n,n+1} \text{ with } x'_i = x_i + 1\}, \\ I^{n,n+1,-}_{adm}(x,y) &= \{1 \leq i \leq n+1 : (x',y) \in W^{n,n+1} \text{ with } x'_i = x_i - 1\}. \end{aligned}$$

We begin, by observing that we have the following time-0 initial condition,

$$\mathsf{q}_0((x,y),(x',y')) = \delta_{(x,y),(x',y')}. \tag{5.32}$$

This follows directly from the form of $\mathsf{q}_t((x,y),(x',y'))$, by noting that as $t \downarrow 0$, the diagonal

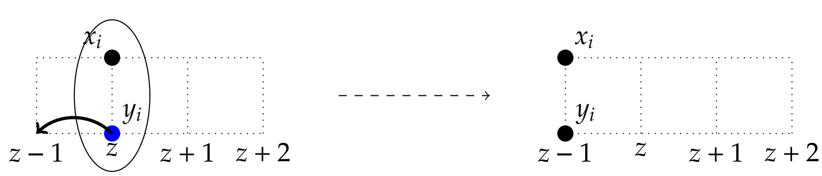


Figure 5.7: In $W^{n,n+1}$, a jump of y_i pushes (induces a simultaneous jump of) x_{i+1} to the left so that the interlacing remains. Here, the jump happens with rate $\hat{\mu}(z) = \lambda(z)$.

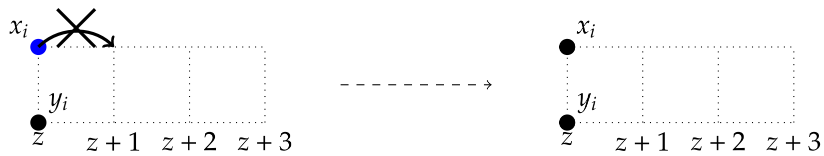


Figure 5.8: In $W^{n,n+1}$, a jump of x_i to the right is blocked by y_i so that the interlacing remains. Here, the clock of x_i rings with rate $\lambda(z)$.

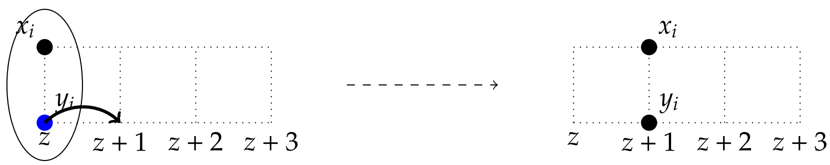


Figure 5.9: In $W^{n,n}$, a jump of y_i pushes (induces a simultaneous jump of) x_i to the right so that the interlacing remains. Here, the jump happens with rate $\lambda(z)$.

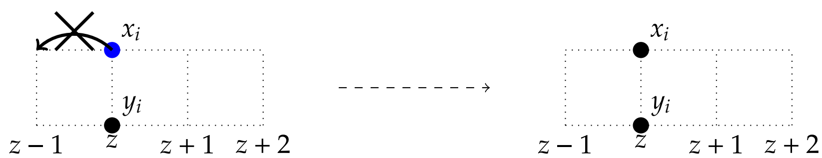


Figure 5.10: In $W^{n,n}$, a jump of x_i to the left is blocked by y_i so that the interlacing remains. Here, the clock of x_i rings with rate $\hat{\mu}(z) = \lambda(z)$.

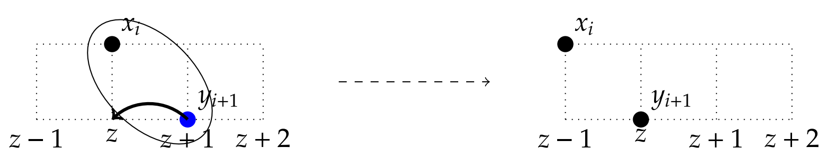


Figure 5.11: In $W^{n,n}$, a jump of y_{i+1} pushes (induces a simultaneous jump of) x_i to the left so that the interlacing remains. Here, the jump happens with rate $\mu(z+1)$.

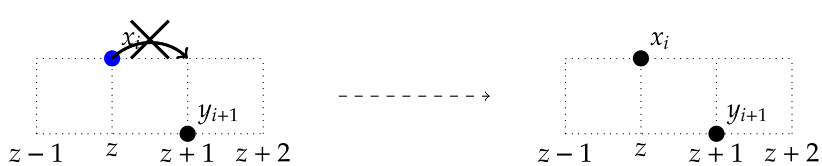


Figure 5.12: In $W^{n,n}$, a jump of x_i to the right is blocked by y_{i+1} so that the interlacing remains. Here, the clock of x_i rings with rate $\hat{\lambda}(z) = \mu(z+1)$.

entries converge to $\delta_{x_i,x'_i}, \delta_{y_i,y'_i}$, while all other contributions to the determinant vanish (or see proof of Proposition 5.11).

Moreover, note that the entries of each matrix in the block determinant $\mathsf{q}_t^{n,n+1}$ namely $\mathsf{A}_t(x,x')$, $\mathsf{B}_t(x,x')$, $\mathsf{C}_t(x,x')$, $\mathsf{D}_t(x,x')$ (we are abusing notation slightly by using the same notation for both the matrices and their scalar entries) solve the following differential equations in the backwards variable x, for any $x, x' \in I$ fixed and $t > 0$,

$$\frac{d}{dt}\mathsf{A}_t(x,x') = \mathcal{D}_x \mathsf{A}_t(x,x'), \tag{5.33}$$

$$\frac{d}{dt}\mathsf{B}_t(x,x') = \mathcal{D}_x \mathsf{B}_t(x,x'), \tag{5.34}$$

$$\frac{d}{dt}\mathsf{C}_t(x,x') = \hat{\mathcal{D}}_x \mathsf{C}_t(x,x'), \tag{5.35}$$

$$\frac{d}{dt}\mathsf{D}_t(x,x') = \hat{\mathcal{D}}_x \mathsf{D}_t(x,x'). \tag{5.36}$$

Observe that the differential equation (5.35) for C_t follows from the Siegmund duality Lemma 5.5 and reversibility with respect to $\hat{\pi}$ of the $\hat{\mathcal{D}}$-chain.

Now, we consider the discrete generator $\mathfrak{D}^{n,n+1}$, the matrix that gives the rates of the push-block dynamics in $W^{n,n+1}$ (see Figures 5.5-5.8 to help visualize the meaning of these rates; also see Remark 5.26 below for the rates in $W^{n,n}$),

$$\mathfrak{D}^{n,n+1}((x,y),(x',y')) = \begin{cases} \lambda(x_i) & x'_i = x_i + 1 \text{ and } i \in I^{n,n+1,+}_{adm}(x,y) \\ \mu(x_i) & x'_i = x_i - 1 \text{ and } i \in I^{n,n+1,-}_{adm}(x,y) \\ \hat{\lambda}(y_i) = \mu(y_i+1) & y'_i = y_i + 1 \text{ and } i+1 \in I^{n,n+1,-}_{adm}(x,y) \\ \hat{\mu}(y_i) = \lambda(y_i) & y'_i = y_i - 1 \text{ and } i \in I^{n,n+1,+}_{adm}(x,y) \\ \hat{\lambda}(y_i) = \mu(y_i+1) & (x_{i+1}, y_i) = (x+1, x), (x'_{i+1}, y'_i) = (x+2, x+1) \\ \hat{\mu}(y_i) = \lambda(y_i) & (x_i, y_i) = (x,x), (x'_i, y'_i) = (x-1, x-1) \\ S^{n,n+1}_{(x,y)} & (x',y') = (x,y) \\ 0 & \text{otherwise} \end{cases},$$

where $S^{n,n+1}_{(x,y)}$ is given by,

$$S^{n,n+1}_{(x,y)} = -\sum_{i \in I^{n,n+1,+}_{adm}(x,y)} \lambda(x_i) - \sum_{i \in I^{n,n+1,-}_{adm}(x,y)} \mu(x_i) - \sum_{i=1}^{n} \left[\hat{\lambda}(y_i) + \hat{\mu}(y_i)\right].$$

Observe that, there is a non-zero rate for the transition $(x,y) \in W^{n,n+1} \to (x',y') \notin W^{n,n+1}$, which corresponds to the chain being killed (in the sequel we will identify all such configurations with a cemetery/absorbing state $\dagger$); this of course coincides with the rate of

$y \in W^n(I) \to y' \notin W^n(I)$, which is non-zero only for $y \in \partial W^n(I)$ and is furthermore given by,

$$k^{n,n+1}_{(x,y)} = \sum_{i=1}^{n-1} \mathbf{1}\left(y_i + 1 = y_{i+1}\right)\left[\hat{\lambda}(y_i) + \hat{\mu}(y_i + 1)\right] + \mathbf{1}\left(y_1 = l\right)\hat{\mu}(l).$$

Moreover, note that the first four conditions, given in terms of the indexing sets $I^{n,n+1,+}_{adm}$ and $I^{n,n+1,-}_{adm}$, in $\mathfrak{D}^{n,n+1}$ above could have been replaced by, $(x', y) \in W^{n,n+1}$ and $(x, y') \in W^{n,n+1}$ respectively. Also, observe that in the definition of $\mathfrak{D}^{n,n+1}$ the first two rates correspond to the free evolution of the X-particles as $\mathcal{D}$-chains, the next two to the evolution of the Y-particles as $\hat{\mathcal{D}}$-chains and the last two to the pushing mechanism (obviously, blocking corresponds to the 0 rate).

Lemma 5.23. *Then, $\mathsf{q}^{n,n+1}_t$ solves the (backwards) differential equation, for $(x, y), (x', y') \in W^{n,n+1}$ and $t > 0$:*

$$\frac{d}{dt}\mathsf{q}^{n,n+1}_t((x, y), (x', y')) = (\mathfrak{D}^{n,n+1}\mathsf{q}^{n,n+1}_t)((x, y), (x', y')).$$

Proof. For $(x, y) \in \mathring{W}^{n,n+1}$, the claim follows immediately from (5.33), (5.34), (5.35), (5.36) and the multilinearity of the determinant. We will hence, now concentrate on the case of $(x, y) \in \partial W^{n,n+1}$. We will only consider the case $x_1 = y_1 = x$, as all others are completely analogous. Moreover, in order to ease notation and make the gist of the simple argument clear we will further restrict our attention to the rows containing x_1, y_1 and in fact it is easy to see that it suffices to consider the 2×2 matrix given by, with $x', y' \in I$ fixed,

$$\det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix}.$$

By taking the $\frac{d}{dt}$-differential of the determinant, we easily see from the differential equations (5.33), (5.34), (5.35), (5.36) that we get,

$$\begin{aligned} \frac{d}{dt}\det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix} = \lambda(x)&\left[\det\begin{pmatrix} \mathsf{A}_t(x+1, x') & \mathsf{B}_t(x+1, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix} - \det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix}\right] \\ +\mu(x)&\left[\det\begin{pmatrix} \mathsf{A}_t(x-1, x') & \mathsf{B}_t(x-1, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix} - \det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix}\right] \\ \mu(x+1)&\left[\det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x+1, x') & \mathsf{D}_t(x+1, y') \end{pmatrix} - \det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix}\right] \\ \lambda(x)&\left[\det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x-1, x') & \mathsf{D}_t(x-1, y') \end{pmatrix} - \det\begin{pmatrix} \mathsf{A}_t(x, x') & \mathsf{B}_t(x, y') \\ \mathsf{C}_t(x, x') & \mathsf{D}_t(x, y') \end{pmatrix}\right]. \end{aligned}$$

On the other hand, what we would like to have, according to the rates of $\mathfrak{D}^{n,n+1}$, is the following,

$$\frac{d}{dt}\det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(x,x') & \mathsf{D}_t(x,y')\end{pmatrix} = \mu(x)\left[\det\begin{pmatrix}\mathsf{A}_t(x-1,x') & \mathsf{B}_t(x-1,y')\\ \mathsf{C}_t(x,x') & \mathsf{D}_t(x,y')\end{pmatrix} - \det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(x,x') & \mathsf{D}_t(x,y')\end{pmatrix}\right]$$
$$\mu(x+1)\left[\det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(x+1,x') & \mathsf{D}_t(x+1,y')\end{pmatrix} - \det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(x,x') & \mathsf{D}_t(x,y')\end{pmatrix}\right]$$
$$\lambda(x)\left[\det\begin{pmatrix}\mathsf{A}_t(x-1,x') & \mathsf{B}_t(x-1,y')\\ \mathsf{C}_t(x-1,x') & \mathsf{D}_t(x-1,y')\end{pmatrix} - \det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(x,x') & \mathsf{D}_t(x,y')\end{pmatrix}\right].$$

We are thus, required to show that,

$$\det\begin{pmatrix}\mathsf{A}_t(x+1,x') & \mathsf{B}_t(x+1,y')\\ \mathsf{C}_t(x,x') & \mathsf{D}_t(x,y')\end{pmatrix} = \det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(x,x') & \mathsf{D}_t(x,y')\end{pmatrix}, \tag{5.37}$$

which corresponds to x_1 being blocked when $x_1 = y_1$ and x_1 tries to jump to the right (see the configuration in Figure 5.8) and also,

$$\det\begin{pmatrix}\mathsf{A}_t(x,x') & \mathsf{B}_t(x,y')\\ \mathsf{C}_t(x-1,x') & \mathsf{D}_t(x-1,y')\end{pmatrix} = \det\begin{pmatrix}\mathsf{A}_t(x-1,x') & \mathsf{B}_t(x-1,y')\\ \mathsf{C}_t(x-1,x') & \mathsf{D}_t(x-1,y')\end{pmatrix}, \tag{5.38}$$

which corresponds to x_1 being pushed to the left when $x_1 = y_1$ and y_1 jumps to the left (see the configuration in Figure 5.7). Observe that, this latter equality in display (5.38) is the same as the one above in display (5.37), after replacing x with $x-1$. Both of these equalities follow from simple row and column operations. First recall,

$$\begin{aligned}\mathsf{A}_t(x,x') &= p_t(x,x') = -\bar{\nabla}_{x'}P_t\mathbf{1}_{[l,x']}(x),\\ \mathsf{B}_t(x,y') &= \hat{\pi}(y')\left(P_t\mathbf{1}_{[l,y']}(x)-1\right),\\ \mathsf{C}_t(y,x') &= \hat{\pi}^{-1}(y)\nabla_y\bar{\nabla}_{x'}P_t\mathbf{1}_{[l,x']}(y),\\ \mathsf{D}_t(y,y') &= -\frac{\hat{\pi}(y')}{\hat{\pi}(y)}\nabla_yP_t\mathbf{1}_{[l,y']}(y) = \hat{p}_t(y,y').\end{aligned}$$

In order to obtain (5.37) and hence (5.38) as well, we work on the RHS and we multiply the second row by $-\hat{\pi}(x)$ and add it to the first row to obtain,

$$\begin{aligned}\mathsf{A}_t(x,x') - \hat{\pi}(x)\mathsf{C}_t(x,x') &= -\bar{\nabla}_{x'}P_t\mathbf{1}_{[l,x']}(x) - \bar{\nabla}_{x'}P_t\mathbf{1}_{[l,x']}(x+1) + \bar{\nabla}_{x'}P_t\mathbf{1}_{[l,x']}(x) = -\bar{\nabla}_{x'}P_t\mathbf{1}_{[l,x']}(x+1)\\ &= \mathsf{A}_t(x+1,x'),\end{aligned}$$

and similarly for the second column, which then gives us the LHS of (5.37). □

We now add a *cemetery state* † to the state space and to (the transition matrix) $\mathsf{q}_t^{n,n+1}$, to make it an honest (i.e. stochastic) transition matrix, denoted by $\tilde{\mathsf{q}}_t^{n,n+1}$. This corresponds to the process with infinite lifetime, that instead of being killed, gets absorbed at † and stays there forever. Observe that, $\dagger = \{(x,y) : y \notin W^n(I)\}$ and so $\tilde{\mathsf{q}}_t^{n,n+1}$ and $\tilde{\mathfrak{D}}^{n,n+1}$ are given by,

$$\begin{aligned}
\tilde{\mathsf{q}}_t^{n,n+1}(z,w) &= \mathsf{q}_t^{n,n+1}(z,w), \quad \text{for } z,w \neq \dagger,\\
\tilde{\mathsf{q}}_t^{n,n+1}(\dagger,w) &= \delta_{\dagger,w},\\
\tilde{\mathsf{q}}_t^{n,n+1}(z,\dagger) &= 1 - \sum_w \mathsf{q}_t^{n,n+1}(z,w)
\end{aligned}$$

and,

$$\begin{aligned}
\tilde{\mathfrak{D}}^{n,n+1}(z,w) &= \mathfrak{D}^{n,n+1}(z,w), \quad \text{for } z,w \neq \dagger,\\
\tilde{\mathfrak{D}}^{n,n+1}(\dagger,w) &= 0, \; w \neq \dagger,\\
\tilde{\mathfrak{D}}^{n,n+1}(z,\dagger) &= \text{ rate of transition: } y \in W^n(I) \to y' \notin W^n(I), \text{ for } z=(x,y)\\
&= k_{(x,y)}^{n,n+1} = \sum_{i=1}^{n-1} \mathbf{1}\left(y_i + 1 = y_{i+1}\right)\left[\hat{\lambda}(y_i) + \hat{\mu}(y_i+1)\right] + \mathbf{1}\left(y_1 = l\right)\hat{\mu}(l).
\end{aligned}$$

Then, from our previous considerations we get:

Proposition 5.24. *For fixed $z,w \in W^{n,n+1} \cup \dagger$ we have for $t>0$,*

$$\frac{d}{dt}\tilde{\mathsf{q}}_t^{n,n+1}(z,w) = (\tilde{\mathfrak{D}}^{n,n+1}\tilde{\mathsf{q}}_t^{n,n+1})(z,w). \tag{5.39}$$

Moreover, $\tilde{\mathsf{q}}_0^{n,n+1} = Id$ and also for $t \geq 0, \tilde{\mathsf{q}}_t^{n,n+1}$ is positive.

We proceed to prove uniqueness of solutions:

Proposition 5.25. *The solution to the backwards equation (5.39) is unique.*

Proof. Following [28] we write $\tilde{\mathfrak{D}}^{n,n+1} = -\mathrm{diag}(\tilde{\mathfrak{D}}^{n,n+1}) + \bar{\mathfrak{D}}^{n,n+1}$ where $\mathrm{diag}(\tilde{\mathfrak{D}}^{n,n+1})(z,w) = -\tilde{\mathfrak{D}}^{n,n+1}(z,w)\mathbf{1}_{zw}$ and $\bar{\mathfrak{D}}^{n,n+1}(z,w) = \hat{\mathfrak{D}}^{n,n+1}(z,w)$ if $z \neq w$ and 0 otherwise. We define the following recursion $\left\{\left(\mathcal{P}^{(k)}(t); t \geq 0\right)\right\}_{k\geq 1}$, of operators (matrices) by, for $t \geq 0$,

$$\begin{aligned}
\mathcal{P}^{(0)}(t) &= e^{-\mathrm{diag}(\tilde{\mathfrak{D}}^{n,n+1})t},\\
\mathcal{P}^{(k)}(t) &= \int_0^t e^{-\mathrm{diag}(\tilde{\mathfrak{D}}^{n,n+1})s}\bar{\mathfrak{D}}^{n,n+1}\mathcal{P}^{(k-1)}(t-s)ds
\end{aligned}$$

and also let $\left(\tilde{\mathcal{P}}(t); t \geq 0\right)$ be given by, for $t \geq 0$,

$$\tilde{\mathcal{P}}(t) = \sum_{k=0}^{\infty} \mathcal{P}^{(k)}(t).$$

Then (see Theorem 4.1, Corollary 4.2 of [28]), $\left(\tilde{\mathcal{P}}(t); t \geq 0\right)$ is the *minimal* solution of the backwards equation, $\frac{d}{dt}S(t) = \tilde{\mathfrak{D}}^{n,n+1}S(t)$ for $t > 0$ and $S(0) = Id$ and if it is *stochastic* then, it is the *unique* one. So, in such a case it must necessarily coincide with $\tilde{\mathsf{q}}_t^{n,n+1}$.

By Proposition 4.3 of [28], in order to show that the minimal solution is indeed stochastic it suffices to prove that for $w \in W^{n,n+1}$, we have $\mathbb{P}_w\left((X(t), Y(t)) \notin w + [-N, N]^{2n+1}\right) \to 0$ as $N \to \infty$, for fixed $t \geq 0$.

Note that,

$$\mathbb{P}_w\left((X(t), Y(t)) \notin w + [-N, N]^{2n+1}\right) \leq 2(n+1)\max\{\mathbb{P}_w\left(X_{n+1}(t) > x_{n+1} + N\right), \\ \mathbb{P}_w\left(X_1(t) < x_1 - N\right)\}.$$

So it suffices to show that the probabilities on the right hand side go to 0 as $N \to \infty$ and since both cases are completely similar, we will show that,

$$\mathbb{P}\left(X_{n+1}(t) > x_{n+1} + N\right)$$

vanishes as $N \to \infty$. This is intuitively obvious, since away from $(Y_n(t); t \geq 0)$, the top particle $(X_{n+1}(t); t \geq 0)$ follows the non-explosive $\mathcal{D}$-chain dynamics and so the only way for it to explode is if Y_n drives it to $+\infty$, which does not happen (since Y_n is itself an autonomous non-exploding $\hat{\mathcal{D}}$-chain). More formally, we have (the notation is made precise below),

$$\mathbb{P}\left(X_{n+1}(t) > x_{n+1} + N\right) \leq \mathbb{E}\left[\mathbb{P}\left(\bar{D}(t) > x_{n+1} + N \middle| \bar{D}(0) = \sup_{s\leq t}\hat{D}(s) \vee x_{n+1}\right)\right]$$

where $\hat{D}$ is a realization of a $\hat{\mathcal{D}}$-chain and the outer expectation is taken over this. Also note that,

$$M = \sup_{s\leq t}\hat{D}(s) < \infty, \text{ a.s.}$$

and conditioned on the realization of $\hat{D}$, the chain $\bar{D}$ is defined as follows: it moves as a $\mathcal{D}$-chain except that, jumps below M are suppressed, namely its rates $(\bar{\lambda}, \bar{\mu})$ are given by,

$$\bar{\lambda}(M) = \lambda(M), \bar{\mu}(M) = 0 \text{ and } \bar{\lambda}(k) = \lambda(k), \bar{\mu}(k) = \mu(k), \text{ for } k \geq M + 1.$$

This is again, non-explosive and hence,

$$\mathbb{P}\left(\bar{D}(t) > x_{n+1} + N \middle| \bar{D}(0) = \sup_{s\leq t}\hat{D}(s) \vee x_{n+1}\right) \to 0, \text{ as } N \to \infty.$$

The result now, follows from the dominated convergence theorem. □

Finally, after a Doob's h-transform, by a strictly positive eigenfunction $\mathfrak{h}$ of $(\hat{P}^n_t; t \geq 0)$, the rates for the two-level Markov process, evolving according to $\left(\mathsf{Q}^{n,n+1,\mathfrak{h}}_t; t \geq 0\right)$ are given by,

$$\mathfrak{D}^{n,n+1}((x,y),(x',y')) = \begin{cases} \lambda(x_i) & x_i' = x_i + 1 \text{ and } i \in I_{adm}^{n,n+1,+}(x,y) \\ \mu(x_i) & x_i' = x_i - 1 \text{ and } i \in I_{adm}^{n,n+1,-}(x,y) \\ \hat{\lambda}_{\mathfrak{h}}^i(y_1,\cdots,y_n) & y_i' = y_i + 1 \text{ and } i+1 \in I_{adm}^{n,n+1,-}(x,y) \\ \hat{\mu}_{\mathfrak{h}}^i(y_1,\cdots,y_n) & y_i' = y_i - 1 \text{ and } i \in I_{adm}^{n,n+1,+}(x,y) \\ \hat{\lambda}_{\mathfrak{h}}^i(y_1,\cdots,y_n) & (x_{i+1},y_i) = (x+1,x), (x_{i+1}',y_i') = (x+2,x+1) \\ \hat{\mu}_{\mathfrak{h}}^i(y_1,\cdots,y_n) & (x_i,y_i) = (x,x), (x_i',y_i') = (x-1,x-1) \\ S_{(x,y)}^{n,n+1,\mathfrak{h}} & (x',y') = (x,y) \\ 0 & \text{otherwise} \end{cases},$$

where for $1 \le i \le n$,

$$\hat{\lambda}_{\mathfrak{h}}^i(y_1,\cdots,y_n) = \frac{\mathfrak{h}(y_1,\cdots,y_{i-1},y_i+1,y_{i+1},\cdots,y_n)}{\mathfrak{h}(y_1,\cdots,y_n)}\hat{\lambda}(y_i),$$
$$\hat{\mu}_{\mathfrak{h}}^i(y_1,\cdots,y_n) = \frac{\mathfrak{h}(y_1,\cdots,y_{i-1},y_i-1,y_{i+1},\cdots,y_n)}{\mathfrak{h}(y_1,\cdots,y_n)}\hat{\mu}(y_i)$$

and $S_{(x,y)}^{n,n+1,\mathfrak{h}}$ is given by,

$$S_{(x,y)}^{n,n+1,\mathfrak{h}} = -\sum_{i \in I_{adm}^{n,n+1,+}(x,y)} \lambda(x_i) - \sum_{i \in I_{adm}^{n,n+1,-}(x,y)} \mu(x_i) - \sum_{i=1}^{n} \left[\hat{\lambda}_{\mathfrak{h}}^i(y_1,\cdots,y_n) + \hat{\mu}_{\mathfrak{h}}^i(y_1,\cdots,y_n)\right].$$

Remark 5.26. *We list here the rates for the push-block dynamics in $W^{n,n}$, described informally in the second paragraph of this subsection. With the analogous (with minor modifications due to the positions of the $\le$ and $<$ signs, see also Figures 5.9-5.12) definitions for $\partial W^{n,n}$, $\mathring{W}^{n,n}$, $I_{adm}^{n,n,+}(x,y)$ and $I_{adm}^{n,n,-}(x,y)$ we have,*

$$\mathfrak{D}^{n,n}((x,y),(x',y')) = \begin{cases} \hat{\lambda}(x_i) & x_i' = x_i + 1 \text{ and } i \in I_{adm}^{n,n,+}(x,y) \\ \hat{\mu}(x_i) & x_i' = x_i - 1 \text{ and } i \in I_{adm}^{n,n,-}(x,y) \\ \lambda(y_i) & y_i' = y_i + 1 \text{ and } i \in I_{adm}^{n,n,-}(x,y) \\ \mu(y_i) & y_i' = y_i - 1 \text{ and } i-1 \in I_{adm}^{n,n,+}(x,y) \\ \lambda(y_i) & (x_i,y_i) = (x,x), (x_i',y_i') = (x+1,x+1) \\ \mu(y_i) & (x_{i-1},y_i) = (x-1,x), (x_{i-1}',y_i') = (x-2,x-1) \\ S_{(x,y)}^{n,n} & (x',y') = (x,y) \\ 0 & \text{otherwise} \end{cases},$$

where $S^{n,n}_{(x,y)}$ is given by,

$$S^{n,n}_{(x,y)} = -\sum_{i \in I^{n,n,+}_{adm}(x,y)} \hat{\lambda}(x_i) - \sum_{i \in I^{n,n,-}_{adm}(x,y)} \hat{\mu}(x_i) - \sum_{i=1}^{n} \left[\lambda(y_i) + \mu(y_i)\right].$$

Again observe that, there is a non-zero rate $(x,y) \in W^{n,n} \to (x',y') \notin W^{n,n}$, which corresponds to killing the chain; this of course coincides with the rate of $y \in W^n(I) \to y' \notin W^n(I)$, which is only non-zero for $y \in \partial W^n(I)$ and is given by,

$$k^{n,n}_{(x,y)} = \sum_{i=1}^{n-1} \mathbf{1}\left(y_i + 1 = y_{i+1}\right)\left[\lambda(y_i) + \mu(y_i+1)\right].$$

The scheme of proof for the fact that $\mathsf{q}^{n,n}_t$ describes the dynamics above is exactly the same as the one followed for $W^{n,n+1}$.

Remark 5.27. *Note that q^{n_1,n_2}_t is the transition kernel of the push-block dynamics in W^{n_1,n_2} starting from **any** initial distribution $\nu(x,y)$, that is supported in W^{n_1,n_2}. One should compare with the "multilevel transition operator" for central or Gibbs measures denoted here by $\mathfrak{A}_t$, considered in Theorem 3.12 of [23] and later used in [46] Proposition 5.3 and [47] section 5.3, that forms a semigroup when restricted to such measures. For the two-level dynamics these correspond to a measure on W^{n_1,n_2} of the form $m_{n_2}(x)\Lambda^{h_{n_1}}_{n_1,n_2}(x,y)$, where m_{n_2} is a measure on W^{n_2} and $\Lambda^{h_{n_1}}_{n_1,n_2}(x,y)$ is a normalized (Markov) intertwining kernel from section 5.2.3. It is of course clear that, $\mathsf{q}^{n_1,n_2,h_{n_1}}_t$ and $\mathfrak{A}_t$ coincide on such measures. Currently, we have no explicit analogue of the transition kernel for at least 3 levels starting from any initial condition.*

5.3.2 Multilevel process construction

Let the state space I, be fixed. Suppose that, we are given a sequence of positive integers, $\mathsf{n}(1) \le \mathsf{n}(2) \le \cdots \le \mathsf{n}(N) \le \cdots$, so that $\mathsf{n}(k) - \mathsf{n}(k-1) \le 1$. Moreover, we have the following (off-diagonal) jump rates (their purpose is explained below),

$$r^+_j : W^{\mathsf{n}(1)} \to \mathbb{R}_+, r^-_j : W^{\mathsf{n}(1)} \to \mathbb{R}_+, \text{ for } 1 \le j \le \mathsf{n}(1),$$
$$\lambda^i : I \to \mathbb{R}_+, \mu^i : I \to \mathbb{R}_+, \text{ for } i \ge 2.$$

For, $k \ge 1$, the k^{th} level will consist of $\mathsf{n}(k)$ (ordered) particles, i.e. will be taking values in $W^{\mathsf{n}(k)}$. We assume that, the rates for the first level, (r^+_j, r^-_j), with $1 \le j \le \mathsf{n}(1)$, which correspond to increasing or decreasing the j^{th}-coordinate by 1 respectively (equivalently the j^{th}-particle jumping to the right or to the left), give rise to non-explosive dynamics in $W^{\mathsf{n}(1)}$. In the setting studied in this work, these are given by a conditioning, using a Doob's h-transformation, of $\mathsf{n}(1)$ independent birth and death chains (see discussion after proof of Proposition 5.25 above for example). Furthermore, assume that the rates $(\lambda^i, \mu^i)_{i \ge 2}$ give rise to non-explosive (one-dimensional) birth and death chains in I.

Our goal is to construct, for each $N \geq 1$, a multilevel interlaced Markov process $\left(X^1(t), \cdots, X^N(t); t \geq 0\right)$ with generator $\mathfrak{D}_{1,\cdots,N}$, such that for each $k \geq 1$, $\left(X^{k+1}(t); t \geq 0\right)$ consists of $\mathsf{n}(k+1)$ independent birth and death chains, each moving with rates $(\lambda^{k+1}, \mu^{k+1})$, pushed and blocked, when at the boundary of $W^{\mathsf{n}(k),\mathsf{n}(k+1)}$ by the (particles of the) process $\left(X^k(t); t \geq 0\right)$, as in our two-level couplings from the previous subsection. We do this by induction. For the first level define,

$$\mathfrak{D}_1\left(x^1, z^1\right) = \begin{cases} r_i^+\left(x^1\right) & z_i^1 = x_i^1 + 1,\ 1 \leq i \leq \mathsf{n}(1) \\ r_i^-\left(x^1\right) & z_i^1 = x_i^1 - 1,\ 1 \leq i \leq \mathsf{n}(1) \\ -\sum_{i=1}^{\mathsf{n}(1)} \left[r_i^+\left(x^1\right) + r_i^-\left(x^1\right)\right] & x^1 = z^1 \\ 0 & \text{otherwise} \end{cases}.$$

Suppose that we have constructed a process $\left(X^1(t), \cdots, X^{N-1}(t); t \geq 0\right)$, with rates of a transition $\left(x^1, \cdots, x^{N-1}\right) \to \left(z^1, \cdots, z^{N-1}\right)$ given by,

$$\mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1, \cdots, x^{N-1}\right), \left(z^1, \cdots, z^{N-1}\right)\right)$$

where, for $i \geq 1$, x^i and x^{i+1}, z^i and z^{i+1}, interlace. We proceed to define the rates $\mathfrak{D}_{1,\cdots,N}$ giving rise to $\left(X^1(t), \cdots, X^N(t); t \geq 0\right)$. First, suppose that $\mathsf{n}(N) = \mathsf{n}(N-1) + 1$. Then we let the jump rates $\left(x^1, \cdots, x^N\right) \to \left(z^1, \cdots, z^N\right)$,

$$\mathfrak{D}_{1,\cdots,N}\left(\left(x^1, \cdots, x^N\right), \left(z^1, \cdots, z^N\right)\right)$$

be given by,

$$\begin{cases} \lambda^N(x_i^N) & z_i^N = x_i^N + 1 \text{ and } i \in I_{adm}^{\mathsf{n}(N)-1,\mathsf{n}(N),+}(x^N, x^{N-1}) \\ \mu^N(x_i^N) & z_i^N = x_i^N - 1 \text{ and } i \in I_{adm}^{\mathsf{n}(N)-1,\mathsf{n}(N),-}(x^N, x^{N-1}) \\ \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1, \cdots, x^{N-1}\right), \left(z^1, \cdots, z^{N-1}\right)\right) & x^N = z^N \text{ and } (x^N, z^{N-1}) \in W^{\mathsf{n}(N)-1,\mathsf{n}(N)} \\ \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1, \cdots, x^{N-1}\right), \left(z^1, \cdots, z^{N-1}\right)\right) & (x_{i+1}^N, x_i^{N-1}) = (x+1, x), (z_{i+1}^N, z_i^{N-1}) = (x+2, x+1) \\ \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1, \cdots, x^{N-1}\right), \left(z^1, \cdots, z^{N-1}\right)\right) & (x_i^N, x_i^{N-1}) = (x, x), (z_i^N, z_i^{N-1}) = (x-1, x-1) \\ S_{1,\cdots,N}^{(x^1,\cdots,x^N)} & \left(x^1, \cdots, x^N\right) = \left(z^1, \cdots, z^N\right) \\ 0 & \text{otherwise} \end{cases},$$

where $S_{1,\cdots,N}^{(x^1,\cdots,x^N)}$ is given by,

$$\begin{aligned} S_{1,\cdots,N}^{(x^1,\cdots,x^N)} = &- \sum_{i \in I_{adm}^{\mathsf{n}(N)-1,\mathsf{n}(N),+}(x^N, x^{N-1})} \lambda^N(x_i^N) - \sum_{i \in I_{adm}^{\mathsf{n}(N)-1,\mathsf{n}(N),-}(x^N, x^{N-1})} \mu^N(x_i^N) \\ &- \sum_{z^1,\cdots,z^{N-1}} \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1, \cdots, x^{N-1}\right), \left(z^1, \cdots, z^{N-1}\right)\right). \end{aligned}$$

Similarly, if $\mathsf{n}(N) = \mathsf{n}(N-1)$ we then define $\mathfrak{D}_{1,\cdots,N}\left(\left(x^1,\cdots,x^N\right),\left(z^1,\cdots,z^N\right)\right)$ as follows,

$$\begin{cases} \lambda^N(x_i^N) & z_i^N = x_i^N + 1 \text{ and } i \in I_{adm}^{\mathsf{n}(N),\mathsf{n}(N),+}(x^N, x^{N-1}) \\ \mu^N(x_i^N) & z_i^N = x_i^N - 1 \text{ and } i \in I_{adm}^{\mathsf{n}(N),\mathsf{n}(N),-}(x^N, x^{N-1}) \\ \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1,\cdots,x^{N-1}\right),\left(z^1,\cdots,z^{N-1}\right)\right) & x^N = z^N \text{ and } (x^N, z^{N-1}) \in W^{\mathsf{n}(N),\mathsf{n}(N)} \\ \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1,\cdots,x^{N-1}\right),\left(z^1,\cdots,z^{N-1}\right)\right) & (x_i^N, x_i^{N-1}) = (x,x), (z_{i+1}^N, z_i^{N-1}) = (x+1, x+1) \\ \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1,\cdots,x^{N-1}\right),\left(z^1,\cdots,z^{N-1}\right)\right) & (x_{i-1}^N, x_i^{N-1}) = (x-1,x), (z_{i-1}^N, z_i^{N-1}) = (x-2, x-1) \\ \tilde{S}_{1,\cdots,N}^{(x^1,\cdots,x^N)} & \left(x^1,\cdots,x^N\right) = \left(z^1,\cdots,z^N\right) \\ 0 & \text{otherwise} \end{cases},$$

where $\tilde{S}_{1,\cdots,N}^{(x^1,\cdots,x^N)}$ is given by,

$$\begin{aligned} \tilde{S}_{1,\cdots,N}^{(x^1,\cdots,x^N)} = &- \sum_{i \in I_{adm}^{\mathsf{n}(N),\mathsf{n}(N),+}(x^N,x^{N-1})} \lambda^N(x_i^N) - \sum_{i \in I_{adm}^{\mathsf{n}(N),\mathsf{n}(N),-}(x^N,x^{N-1})} \mu^N(x_i^N) \\ &- \sum_{z^1,\cdots,z^{N-1}} \mathfrak{D}_{1,\cdots,N-1}\left(\left(x^1,\cdots,x^{N-1}\right),\left(z^1,\cdots,z^{N-1}\right)\right). \end{aligned}$$

Observe that, by construction for any $1 \le k \le N$, the process consisting of the first k levels, $\left(X^1(t),\cdots,X^k(t); t \ge 0\right)$ is autonomous, governed by the transition rates $\mathfrak{D}_{1,\cdots,k}$. Moreover, given the trajectories of $\left(X^k(t); t \ge 0\right)$, the very next $(k+1)^{st}$ level $\left(X^{k+1}(t); t \ge 0\right)$, simply moves according to the corresponding push-block dynamics in either $W^{\mathsf{n}(k),\mathsf{n}(k)+1}$ or $W^{\mathsf{n}(k),\mathsf{n}(k)}$.

The fact that, the process with transition matrix $\mathfrak{D}_{1,\cdots,N}$ just defined, is well-posed can be seen inductively as follows. Assume that $\left(X^1(t),\cdots,X^{N-1}(t); t \ge 0\right)$ is almost surely non-explosive. Then by definition, adding level-N, $\left(X^N(t); t \ge 0\right)$ means introducing $\mathsf{n}(N)$ further independent birth and death chains (particles) each moving according to the non-explosive jump rates (λ^N, μ^N) that only interact with $\left(X^{N-1}(t); t \ge 0\right)$ via the pushing and blocking mechanism. Hence, this new enlarged process is seen to be non-explosive by the exact same argument used at the end of the preceding subsection.

5.3.3 Consistent dynamics for multilevel processes

We will discuss consistency relations under which if the multilevel process, whose construction we have just described, is started according to certain *Gibbs* or *central* initial conditions, then each level evolves as a Markov process and the fixed time $T > 0$ distribution of the whole process retains the explicit Gibbs structure (see Section 1.3 for the analogous construction in the diffusion setting). We restrict our attention to multilevel processes taking values in triangular arrays known as Gelfand-Tsetlin patterns. The consistency relations and Propositions 5.28 and 5.30 below have analogues, with rather obvious modifications, to arbitrary multilevel interlaced processes, so that the number of particles from one level to the next increases by at most 1. We do not spell this out, since the already heavy notation

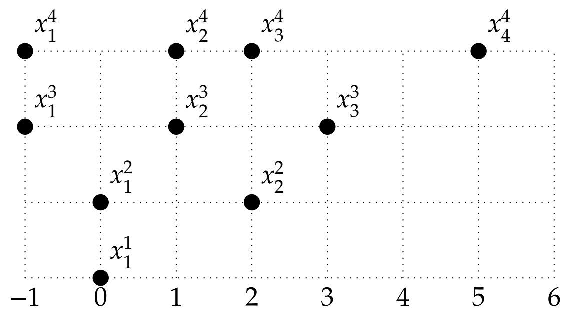


Figure 5.13: An example of a Gelfand-Tsetlin pattern of depth 4 for $I = \mathbb{Z}$, with $x^1 = 0, x^2 = (0,2), x^3 = (-1,1,3), x^4 = (-1,1,2,5)$.

becomes quite cumbersome.

Before we continue, we note that none of the results of this subsection are essentially new. In recent years Borodin and collaborators have many variations of constructions of multilevel processes (see Remark 5.31). In particular Propositions 5.28 and 5.30 follow as corollaries, after setting things up carefully, of the results found in Section 8 of [28] (see also Section 9 therein). The reader familiar with those constructions can safely skip to the statements of the propositions (or skip the current subsection altogether). The reason we decided to include this rather detailed section, other than for completeness of the chapter and sake of exposition, is because our method of proof is different; in particular the explicit form of the transition kernel of the two-level dynamics (cf. Remark 5.27) does not appear in any of those works (and is special to our setting).

We first consider the Gelfand-Tsetlin patterns of type-A, with N levels. These are defined as follows,

$$\mathbb{GT}(N) = \left\{ \left(x^1, \cdots, x^N\right) : x^i \in W^{i,i+1}(x^{i+1}), \text{ for } 1 \leq i \leq N-1 \right\}. \tag{5.40}$$

See Figure 5.13 for an example.

Suppose we have, for $1 \leq k \leq N$, rates $(\lambda^k(\cdot), \mu^k(\cdot))$ governing modulo interactions the k independent birth and death chains of the k^{th} level. Denote by, $p_t^k(\cdot,\cdot)$ the transition density of this chain, also let $\hat{p}_t^k(\cdot,\cdot)$ be the transition density and $\hat{\pi}^k(\cdot)$ the symmetrizing measure of its Siegmund dual chain (with rates $(\hat{\lambda}^k(\cdot), \hat{\mu}^k(\cdot))$. Finally, with these rates as input, construct the process $\left(X^1(t), \cdots, X^N(t); t \geq 0\right)$ via the procedure detailed in subsection 5.3.2 above.

We want to be able to apply Proposition 5.19 (and Theorem 5.18) repeatedly recursively, for $k \geq 2$, to each pair $\left(X^{k-1}, X^k\right)$. Towards this end, suppose X^{k-1} is distributed as a Markov process in W^{k-1}, evolving according to the Doob's h-transformed Karlin-McGregor semigroup, by the strictly positive eigenfunction h_{k-1}, with eigenvalue $e^{c_{k-1}t}$, having transition density,

$$e^{-c_{k-1}t}\frac{h_{k-1}(y_1,\cdots,y_{k-1})}{h_{k-1}(x_1,\cdots,x_{k-1})}\det\left(\hat{p}^k_t\left(x_i,y_j\right)\right)^{k-1}_{i,j=1}.$$

Moreover, define for $k\geq 2$ the following strictly positive function on W^k,

$$H_{k-1}\left(x_1,\cdots,x_k\right)=\sum_{y\in W^{k-1,k}(x)}\prod_{i=1}^{k-1}\hat{\pi}^k(y_i)h_{k-1}(y_1,\cdots,y_{k-1}). \tag{5.41}$$

Then, the basic consistency relation at the level of transition densities, which guarantees that the two descriptions of X^k as the non-autonomous component of the coupling $\left(X^{k-1},X^k\right)$ and the autonomous component of the coupling $\left(X^k,X^{k+1}\right)$ match, becomes for $k\geq 2$,

$$e^{-c_{k-1}t}\frac{H_{k-1}(y_1,\cdots,y_k)}{H_{k-1}(x_1,\cdots,x_k)}\det\left(p^k_t\left(x_i,y_j\right)\right)^{k}_{i,j=1}=e^{-c_kt}\frac{h_k(y_1,\cdots,y_k)}{h_k(x_1,\cdots,x_k)}\det\left(\hat{p}^{k+1}_t\left(x_i,y_j\right)\right)^{k}_{i,j=1}. \tag{5.42}$$

For $k=1$ we put by definition $H_0\equiv 1$ and so,

$$p^1_t\left(x,y\right)=e^{-c_1t}\frac{h_1(y)}{h_1(x)}\hat{p}^2_t\left(x,y\right). \tag{5.43}$$

Let $\left(\mathfrak{P}^k(t);t\geq 0\right)$, denote the Markov semigroup that these densities give rise to and also define the Markov kernel,

$$\mathfrak{L}^k_{k-1}\left(x,y\right)=\frac{\prod_{i=1}^{k-1}\hat{\pi}^k(y_i)h_{k-1}(y_1,\cdots,y_{k-1})}{H_{k-1}(x_1,\cdots,x_k)}\mathbf{1}\left(y\in W^{k-1,k}(x)\right).$$

Then, we have the following proposition.

Proposition 5.28. *Let $\left(X^1(t),\cdots,X^N(t);t\geq 0\right)$ be the Markov process with transition matrix $\mathfrak{D}_{1,\cdots,N}$, built from the non-explosive rates $(\lambda^i(\cdot),\mu^i(\cdot))_{1\leq i\leq N}$. Suppose the consistency relations (5.41) and (5.42) hold for $1\leq k\leq N-1$. Let $\left(\mathfrak{P}^k(t);t\geq 0\right)$ and $\mathfrak{L}^k_{k-1}$ denote the semigroups and Markov kernels defined above and let $\mathfrak{M}^N(\cdot)$ be a probability measure on W^N. Finally, suppose that, $\left(X^1(t),\cdots,X^N(t);t\geq 0\right)$ is initialized according to the Gibbs measure with density in $\mathbb{GT}(N)$,*

$$\mathfrak{M}^N(x^N)\mathfrak{L}^N_{N-1}\left(x^N,x^{N-1}\right)\cdots\mathfrak{L}^2_1\left(x^2,x^1\right). \tag{5.44}$$

Then, $\left(X^k(t);t\geq 0\right)$ for $1\leq k\leq N$ is distributed as a Markov process evolving according to $\left(\mathfrak{P}^k(t);t\geq 0\right)$ and moreover, for fixed $T>0$, the law of $\left(X^1(T),\cdots,X^N(T)\right)$ is given by the evolved Gibbs measure, with density in $\mathbb{GT}(N)$,

$$\left[\mathfrak{M}^N\mathfrak{P}^N(T)\right](x^N)\mathfrak{L}^N_{N-1}\left(x^N,x^{N-1}\right)\cdots\mathfrak{L}^2_1\left(x^2,x^1\right). \tag{5.45}$$

Proof. The proof is by induction. For $N=2$, this is Proposition 5.19 (see Theorem 5.18 as well). Assume the result is true for $N-1$. Then, $\left(X^{N-1}(t); t \geq 0\right)$ is a Markov process with semigroup $\left(\mathfrak{P}^{N-1}(t); t \geq 0\right)$. Moreover, from the consistency relation (5.42) for $k = N-1$, the joint dynamics of $\left(X^{N-1}(t), X^{N}(t); t \geq 0\right)$ are those considered in Proposition 5.19 and thus, we obtain that $\left(X^{N}(t); t \geq 0\right)$ is distributed as a Markov process with semigroup $\left(\mathfrak{P}^{N}(t); t \geq 0\right)$. Furthermore, for fixed $T > 0$, the conditional law of $X^{N-1}(T)$ given $X^{N}(T)$ is $\mathfrak{L}^{N}_{N-1}\left(X^{N}(T), \cdot\right)$. Hence, since the distribution of $X^{N}(T)$ has density $\left[\mathfrak{M}^{N}\mathfrak{P}^{N}(T)\right](\cdot)$, we get by the induction hypothesis, that the fixed time $T > 0$, distribution of $\left(X^{1}(T), \cdots, X^{N}(T)\right)$ is given by (5.45). $\square$

Remark 5.29. *If there exist (positive) functions $\{f_k(\cdot)\}_{k=2}^{N}$ such that, for $2 \leq k \leq N$,*

$$h_k(x_1, \cdots, x_k) = \prod_{i=1}^{k} f_k(x_i) H_{k-1}(x_1, \cdots, x_k)$$

and moreover functions $\{G_k(T, \cdot)\}_{k=1}^{N}$ so that,

$$\left[\mathfrak{M}^{N}\mathfrak{P}^{N}(T)\right](x_1, \cdots, x_N) = H_{N-1}(x_1, \cdots, x_N) \det\left(G_i(T, x_j)\right)_{i,j=1}^{N}$$

then (5.45) simplifies to:

$$\det\left(G_i\left(T, x_j^{N}\right)\right)_{i,j=1}^{N} \prod_{k=2}^{N}\prod_{i=1}^{k-1} \hat{\pi}^{k}\left(x_i^{k-1}\right) h_1\left(x_1^{1}\right) \prod_{k=2}^{N}\prod_{i=1}^{k} f_k\left(x_i^{k}\right) \prod_{k=1}^{N-1} \mathbf{1}\left(x^{k} \in W^{k,k+1}(x^{k+1})\right).$$

Hence, since the interlacing constraints can be written as a determinant, for some function $g(\cdot, \cdot)$ of two variables, see section 5.7 for the details, the display above becomes,

$$\det\left(G_i\left(T, x_j^{N}\right)\right)_{i,j=1}^{N} \prod_{k=2}^{N}\prod_{i=1}^{k-1} \hat{\pi}^{k}\left(x_i^{k-1}\right) h_1\left(x_1^{1}\right) \prod_{k=2}^{N}\prod_{i=1}^{k} f_k\left(x_i^{k}\right) \prod_{k=1}^{N-1} \det\left(g\left(x_i^{k}, x_j^{k+1}\right)\right)_{i,j=1}^{k+1}.$$

These types of measures, by the celebrated Eynard-Mehta Theorem (see [35]), give rise to determinantal point processes with an extended correlation kernel K*, which can in principle be computed.*

In order to obtain this explicitly however, one has to invert a certain matrix or do some kind of bi-orthogonalization which is usually a very daunting task. For a particular, but still quite general, solution of the consistency relations, in the setting of a symplectic Gelfand-Tsetlin pattern, see the discussion after Proposition 5.30 below, we are able to perform such a computation in Section 5.10 later on. In fact these computations carry over to a large class of consistent probability measures, that include the ones corresponding to the dynamics considered in this section as special cases, the reader is referred to sections 5.8 to 5.10 for these developments.

We shall now consider coherent dynamics in symplectic Gelfand-Tsetlin patterns of depth N defined by,

$$\mathbb{GT}_{\mathbf{s}}(N) = \left\{\left((x^{(0,1)}, x^{(1,1)} \cdots, x^{(N-1,N)}\right) : x^{(i-1,i)} \in W^{i,i}(x^{(i,i)}), x^{(i,i)} \in W^{i,i+1}(x^{(i,i+1)})\right\}, \tag{5.46}$$

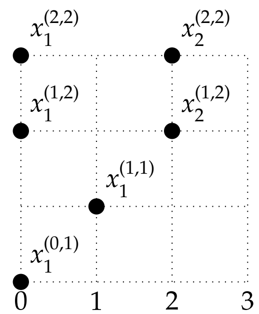


Figure 5.14: An example of a symplectic Gelfand-Tsetlin pattern of depth 2 (note that it has 4 levels), for $I = \mathbb{N}$, with $x^{(0,1)} = 0, x^{(1,1)} = 1, x^{(1,2)} = (0,2), x^{(2,2)} = (0,2)$.

with the notation convention of using two superscript indices to indicate the number of particles at both the preceding and current levels. See Figure 5.14 for a simple example.

Suppose that, for each level of $\mathbb{GT}_{\mathbf{s}}(N)$ we are given (non-explosive) birth and death rates $(\lambda^{(k,k)}(\cdot), \mu^{(k,k)}(\cdot))$ and $(\lambda^{(k,k+1)}(\cdot), \mu^{(k,k+1)}(\cdot))$ and from these we construct a Markov process $\left(X^{(0,1)}(t), X^{(1,1)}(t) \cdots, X^{(N-1,N)}(t); t \geq 0\right)$, using the recipe detailed in subsection 5.3.2. In order to proceed and be able to state the basic consistency relations, we need one more piece of notation. Define the operation $\check{}$ on transition matrices of birth and death (or bilateral) chains, as the inverse of the $\hat{}$ operation, i.e. as the inverse of taking the Siegmund dual. More explicitly, for a chain with birth rates $b(\cdot)$ and death rates $d(\cdot)$ this is given by:

$$\left(\check{b}(z), \check{d}(z)\right) \overset{\text{def}}{=} (d(z), b(z-1)), \; z \in I.$$

Observe that, in case $I = \mathbb{N}$ this is only defined on chains absorbed at -1. Finally, we shall use the same notations as before, with obvious modifications, for the transition densities and symmetrizing measures of the chains with rates $(\lambda^{(k,k)}(\cdot), \mu^{(k,k)}(\cdot))$, $(\lambda^{(k,k+1)}(\cdot), \mu^{(k,k+1)}(\cdot))$ and their various transforms.

We would like Proposition 5.19 (see also Theorem 5.18) to be applicable, for $1 \leq k \leq N-1$, to each pair $\left(X^{(k,k)}, X^{(k,k+1)}\right)$ and Theorem 5.20 to be applicable, for $1 \leq k \leq N-1$, to each pair of the form $\left(X^{(k-1,k)}, X^{(k,k)}\right)$, respectively.

Towards this end, suppose that $X^{(k-1,k-1)}$ evolves according to the h-transformed, by the strictly positive eigenfunction $h_{k-1,k-1}$ with eigenvalue $e^{c_{k-1,k-1}t}$, Karlin-McGregor semi-group with transition kernel in W^{k-1},

$$e^{-c_{k-1,k-1}t}\frac{h_{k-1,k-1}(y_1, \cdots, y_{k-1})}{h_{k-1,k-1}(x_1, \cdots, x_{k-1})} \det\left(\hat{p}_t^{(k-1,k)}\left(x_i, y_j\right)\right)_{i,j=1}^{k-1}$$

and moreover, define for $k \geq 2$ the following strictly positive function on W^k,

$$H_{k-1,k-1}(x_1, \cdots, x_k) = \sum_{y \in W^{k-1,k}(x)} \prod_{i=1}^{k-1} \hat{\pi}^{(k-1,k)}(y_i) h_{k-1,k-1}(y_1, \cdots, y_{k-1}). \tag{5.47}$$

We also define, $H_{0,0} \equiv 1$. Similarly, suppose that $X^{(k-1,k)}$ evolves according to the following h-transformed, by the strictly positive eigenfunction $h_{k-1,k}$ with eigenvalue $e^{c_{k-1,k}t}$, Karlin-McGregor semigroup with transition kernel in W^k,

$$e^{-c_{k-1,k}t}\frac{h_{k-1,k}(y_1,\cdots,y_k)}{h_{k-1,k}(x_1,\cdots,x_k)}\det\left(\check{p}_t^{(k,k)}\left(x_i,y_j\right)\right)_{i,j=1}^{k}$$

and also, define for $k \geq 1$ the following strictly positive function on W^k,

$$H_{k-1,k}\left(x_1,\cdots,x_k\right)=\sum_{y\in W^{k,k}(x)}\prod_{i=1}^{k}\check{\pi}^{(k,k)}(y_i)h_{k-1,k}(y_1,\cdots,y_k). \tag{5.48}$$

Then, the basic consistency relations at the level of transition densities, which ensure that the descriptions of the levels $X^{(k-1,k)}$ and $X^{(k,k)}$ in two consecutive two-level couplings match, become,

$$e^{-c_{k-1,k-1}t}\frac{H_{k-1,k-1}(y_1,\cdots,y_k)}{H_{k-1,k-1}(x_1,\cdots,x_k)}\det\left(p_t^{(k-1,k)}\left(x_i,y_j\right)\right)_{i,j=1}^{k}=e^{-c_{k-1,k}t}\frac{h_{k-1,k}(y_1,\cdots,y_k)}{h_{k-1,k}(x_1,\cdots,x_k)}\det\left(\check{p}_t^{(k,k)}\left(x_i,y_j\right)\right)_{i,j=1}^{k}, \tag{5.49}$$

$$e^{-c_{k-1,k}t}\frac{H_{k-1,k}(y_1,\cdots,y_k)}{H_{k-1,k}(x_1,\cdots,x_k)}\det\left(p_t^{(k,k)}\left(x_i,y_j\right)\right)_{i,j=1}^{k}=e^{-c_{k,k}t}\frac{h_{k,k}(y_1,\cdots,y_k)}{h_{k,k}(x_1,\cdots,x_k)}\det\left(\hat{p}_t^{(k,k+1)}\left(x_i,y_j\right)\right)_{i,j=1}^{k}. \tag{5.50}$$

Let, $\left(\mathfrak{P}^{(k-1,k)}(t);t\geq 0\right)$ and $\left(\mathfrak{P}^{(k,k)}(t);t\geq 0\right)$ denote the corresponding semigroups these transition densities give rise to and finally define the Markov kernels,

$$\mathfrak{L}^{(k-1,k)}(x,y)=\frac{\prod_{i=1}^{k-1}\hat{\pi}^{(k-1,k)}(y_i)h_{k-1,k-1}(y_1,\cdots,y_{k-1})}{H_{k-1,k-1}(x_1,\cdots,x_k)}\mathbf{1}\left(y\in W^{k-1,k}(x)\right),$$

$$\mathfrak{L}^{(k,k)}(x,y)=\frac{\prod_{i=1}^{k}\check{\pi}^{(k,k)}(y_i)h_{k-1,k}(y_1,\cdots,y_k)}{H_{k-1,k}(x_1,\cdots,x_k)}\mathbf{1}\left(y\in W^{k,k}(x)\right).$$

Then, with similar considerations as in Proposition 5.28 above, by inductively applying Proposition 5.19 and Theorem 5.20 interchangeably we obtain:

Proposition 5.30. *Let* $\left(X^{(0,1)}(t),X^{(1,1)}(t)\cdots,X^{(N-1,N)}(t);t\geq 0\right)$ *be the multilevel Markov process in* $\mathbb{GT}_{\mathsf{s}}(N)$ *built from the (non-explosive) rates* $(\lambda^{(k,k)}(\cdot),\mu^{(k,k)}(\cdot))$ *and* $(\lambda^{(k,k+1)}(\cdot),\mu^{(k,k+1)}(\cdot))$. *Suppose that, for all k the consistency relations (5.50) hold. Let* $\mathfrak{M}^{(N-1,N)}(\cdot)$ *be a probability measure on* W^N. *Suppose that,* $\left(X^{(0,1)}(t),X^{(1,1)}(t)\cdots,X^{(N-1,N)}(t);t\geq 0\right)$ *is initialized according to the Gibbs measure with density in* $\mathbb{GT}_{\mathsf{s}}(N)$,

$$\mathfrak{M}^{(N-1,N)}(x^{(N-1,N)})\mathfrak{L}_{N-1}^{N}\left(x^{(N-1,N)},x^{(N-1,N-1)}\right)\cdots\mathfrak{L}_1^2\left(x^{(1,2)},x^{(1,1)}\right)\mathfrak{L}_1^1\left(x^{(1,1)},x^{(0,1)}\right). \tag{5.51}$$

Then, for each k the projections $\left(X^{(k,k)}(t);t\geq 0\right)$ *and* $\left(X^{(k,k+1)}(t);t\geq 0\right)$ *are distributed as Markov processes, evolving according to the semigroups* $\left(\mathfrak{P}^{(k,k)}(t);t\geq 0\right)$ *and* $\left(\mathfrak{P}^{(k,k+1)}(t);t\geq 0\right)$ *respectively.*

Moreover, for fixed times $T > 0$, the law of $\left(X^{(0,1)}(T), X^{(1,1)}(T)\cdots, X^{(N-1,N)}(T)\right)$ has density in $\mathbb{GT}_s(N)$ given by,

$$\left[\mathfrak{M}^{(N-1,N)}\mathfrak{P}^{(N-1,N)}(T)\right](x^{(N-1,N)})\mathfrak{L}_{N-1}^{N}\left(x^{(N-1,N)}, x^{(N-1,N-1)}\right)\cdots\mathfrak{L}_1^2\left(x^{(1,2)}, x^{(1,1)}\right)\mathfrak{L}_1^1\left(x^{(1,1)}, x^{(0,1)}\right). \tag{5.52}$$

The most natural solution (this fact is readily checked) to the consistency relations (5.49) and (5.50) in a symplectic Gelfand-Tsetlin pattern, for $I = \mathbb{N}$, is given by, with $(\lambda(\cdot), \mu(\cdot))$ being the rates of a reflecting at the origin (non-exploding) birth and death chain,

$$\left(\lambda^{(k,k+1)}(\cdot), \mu^{(k,k+1)}(\cdot)\right) = (\lambda(\cdot), \mu(\cdot)), \text{ for } k \geq 0, \tag{5.53}$$

$$\left(\lambda^{(k,k)}(\cdot), \mu^{(k,k)}(\cdot)\right) = \left(\hat{\lambda}(\cdot), \hat{\mu}(\cdot)\right), \text{ for } k \geq 1. \tag{5.54}$$

As already stated several times, this particular construction and its intimate relation to orthogonal polynomials will be studied in detail in later sections.

Remark 5.31. *As already mentioned, a related approach for constructing continuous-time consistent multivariate/multilevel dynamics on countable spaces, which partly inspired our exposition, can be found in Section 8 of [28]. This takes as input the following: a sequence $E_1, \cdots, E_N$ of countable sets, $Q_1, \cdots, Q_N$ (regular) matrices of transition rates on these sets (equivalently $(P_1(t); t \geq 0), \cdots, (P_N(t); t \geq 0)$ the Markovian semigroups corresponding to them) and Markov kernels $\Lambda_1^2, \cdots, \Lambda_{N-1}^N$:*

$$\Lambda_{k-1}^k : E_k \times E_{k-1} \to [0,1]\,, \sum_{y \in E_{k-1}} \Lambda_{k-1}^k(x,y) = 1, \forall x \in E_k,\ k = 2, \cdots, N.$$

Finally, it is assumed that the intertwining/coherency relations between the (single level) semigroups/transition matrices hold, for $k = 2, \cdots, N$:

$$\begin{aligned} Q_k\Lambda_{k-1}^k &= \Lambda_{k-1}^k Q_{k-1}, \\ P_k(t)\Lambda_{k-1}^k &= \Lambda_{k-1}^k P_{k-1}(t),\ t \geq 0. \end{aligned}$$

Then, from this data a consistent coupling is provided, with the analogous consequences of Proposition 5.28 and 5.30 above, see Proposition 8.6 in [28]. In particular, using only the single level intertwining relations (5.29) and (5.31), which are elementary to obtain c.f. Remark 5.21, we could have made use of the theory developed in Section 8 of [28] to construct consistent multilevel dynamics. However, since we already have a two-level coupling, from which as we tried to stress throughout this work (5.29) and (5.31) originate after all, and for completeness of this thesis, we decided to present and discuss in detail the multilevel construction in subsections 5.3.2 and 5.3.3.

5.4 Branching graphs and Markov processes on their boundaries

5.4.1 General setup of branching graphs

We assume that we are given a set of vertices V, decomposed into levels $V = \sqcup_{N=1}^{\infty} V_N$, where each V_N is countable. We moreover, assume that for each $x \in V_{N+1}$ there is at least one edge but not infinitely many connecting it to a vertex in V_N and for each $y \in V_N$ there is at least one edge connecting it to a vertex in V_{N+1}. There are no edges between vertices of non-consecutive levels.

For $N \geq 1$ and each $x \in V_{N+1}$ and $y \in V_N$, let $\mathsf{mult}(x,y) \in \mathbb{R}_+$ denote the multiplicity or weight of the edge connecting x and y. If there is no such edge then this is 0. Define inductively the dimension of $x \in V_{N+1}$ by,

$$\mathsf{dim}_{N+1}(x) = \sum_{y \in V_N} \mathsf{mult}(x,y)\mathsf{dim}_N(y).$$

Note that, we need to stipulate $\mathsf{dim}_1(\cdot)$ for vertices at the first level. In all the examples that we consider, this will always be 1. We can then define the Markov kernel or link $\Lambda_N^{N+1} : V_{N+1} \to V_N$ (note that this is a generalized map, that maps a point in V_{N+1} to a probability measure on V_N) as follows,

$$\Lambda_N^{N+1}(x,y) = \frac{\mathsf{mult}(x,y)\mathsf{dim}_N(y)}{\mathsf{dim}_{N+1}(x)}.$$

We will now precise the notion of the boundary of the graph; this fits into the general abstract framework of projective systems of measures (see Section 3.2.2) although our exposition here will be slightly different (following [120], special to the discrete setting). Denoting by $\mathcal{M}_p(E)$ the space of probability measures on a measurable space E ($\mathcal{M}_p(E)$ is a Banach space with the total variation norm), the kernels $\{\Lambda_N^{N+1}\}_{N\geq 1}$ induce the following projective chain,

$$\mathcal{M}_p(V_1) \leftarrow \mathcal{M}_p(V_2) \leftarrow \cdots \mathcal{M}_p(V_N) \leftarrow \cdots .$$

The projective limit $\varprojlim \mathcal{M}_p(V_N)$, is by definition the convex set consisting of sequences of probability measures $\{\mu_N\}_{N=1}^{\infty}$ that are coherent with respect to the links (in the language of Section 3.2.2 the inverse system of simplices),

$$\mu_{N+1}\Lambda_N^{N+1} = \mu_N,$$

or more explicitly, for $y \in V_N$,

$$\mu_N(y) = \sum_{x \in V_{N+1}} \mu_{N+1}(x)\Lambda_N^{N+1}(x,y).$$

This space is equipped with the projective limit topology. Now, we will call the *extreme points* of $\varprojlim \mathcal{M}_p(V_N)$ denoted by $V_\infty = \mathrm{Ex}\left(\varprojlim \mathcal{M}_p(V_N)\right)$, the *boundary* of the branching graph (or more generally of the projective chain) with the topology inherited from $\varprojlim \mathcal{M}_p(V_N)$. Then, from Theorem 9.2 of [120] (see also [168]) we get that if $V_\infty \neq 0$, then there exists a natural map,

$$\mathcal{M}_p(V_\infty) \to \varprojlim \mathcal{M}_p(V_N), \tag{5.55}$$

that is an isomorphism of measurable spaces. More precisely, V_∞ comes along with a family of (abstract) Markov kernels $\Lambda_N^\infty : V_\infty \to V_N$, which induce a map $\mathcal{M}_p(V_\infty) \to \varprojlim \mathcal{M}_p(V_N)$, which is an isomorphism of measurable spaces. It is a remarkable fact that in certain concrete situations the (abstract) Markov kernels $\Lambda_N^\infty : V_\infty \to V_N$ can be given explicitly. Moreover, we will say that a Markov kernel from a locally compact space X to a locally compact space Y is Feller if the induced contraction that maps $C(Y)$ to $C(X)$ in fact maps $C_0(Y)$ into $C_0(X)$, the continuous functions vanishing at infinity. We finally come to the following definition.

We shall say that, V_∞ is the *Feller boundary* of the branching graph if V_∞ is locally compact, for all $N \geq 1$ the Markov kernels Λ_N^{N+1},Λ_N^∞ are Feller and furthermore the map (5.55) is an isomorphism of measurable spaces.

5.4.2 Method of intertwiners and semigroups on the boundary

The following theorem, stated in the special setting of branching graphs (see also Section 3.2.3), is known as the method of intertwiners, first proven by Borodin and Olshanski in [28]:

Theorem 5.32. *Assume that V_∞ is the Feller boundary of the branching graph described above. Assume that, $\forall N \geq N_0$ we have Feller semigroups $(P_N(t); t \geq 0)$ on the levels V_N, that satisfy the following intertwining relations, for all $t \geq 0$ and $N \geq N_0$,*

$$P_{N+1}(t)\Lambda_N^{N+1} = \Lambda_N^{N+1} P_N(t).$$

Then, there exists a unique Feller semigroup $(P_\infty(t); t \geq 0)$ on V_∞ such that,

$$P_\infty(t)\Lambda_N^\infty = \Lambda_N^\infty P_N(t), \text{ for } t \geq 0, N \geq N_0.$$

Furthermore, if μ_N is the unique invariant probability measure for $(P_N(t); t \geq 0)$ then there exists a unique probability measure μ_∞ on V_∞ that is invariant with respect to $(P_\infty(t); t \geq 0)$.

5.4.3 Examples of branching graphs

In this subsection, we describe three examples of branching graphs. The first two are classical and originated from the representation theory of Lie groups. The third one, the

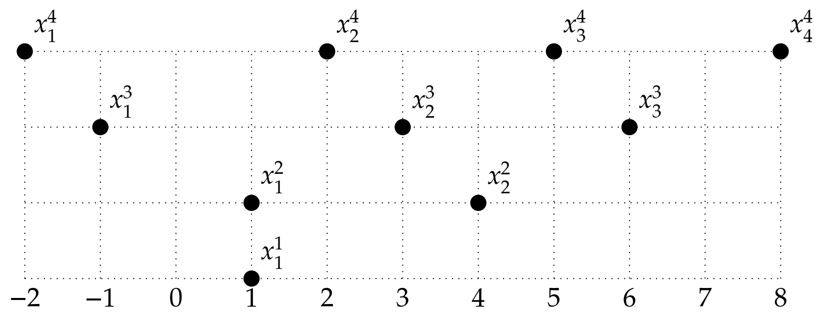


Figure 5.15: An example of a path of length 4 in the Gelfand-Tsetlin graph, given by a Gelfand-Tsetlin pattern of depth 4. Here the path in terms of signatures $\kappa^1 \to \kappa^2 \to \kappa^3 \to \kappa^4$ is given by $\kappa^1 = 1, \kappa^2 = (3,1), \kappa^3 = (4,2,-1), \kappa^4 = (5,3,1,-2)$, which transformed into our notation gives, $x^1 = 1, x^2 = (1,4), x^3 = (-1,3,6), x^4 = (-2,2,5,8)$.

generalized BC-type branching graph, is new and is related to the two-step branching rules for the multivariate Karlin-McGregor polynomials. We will provide rather complete information for the Gelfand-Tsetlin graph, since we will mainly focus on it in Section 5.5. The same kind of information is available for the BC-type graph, although the notation gets a bit more cumbersome, while for the generalized BC-type branching graph much less is known.

The Gelfand-Tsetlin graph The vertices at level N of this branching graph are given by *signatures* of length N, i.e. integer sequences $\kappa = (\kappa_1, \cdots, \kappa_N)$ so that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N$. Moreover, vertices κ at level N and ν at level $N+1$ are connected if they interlace in the following way, $\nu_1 \geq \kappa_1 \geq \nu_2 \geq \cdots \geq \kappa_N \geq \nu_{N+1}$, the multiplicity $\mathsf{mult}(\nu, \kappa)$ being equal to 1 in such a case. To transform this into our notation, note that there is a bijection,

$$(\kappa_1 \geq \cdots \geq \kappa_N) \mapsto (y_1 < y_2 < \cdots < y_N),$$

given by,

$$\tilde{\kappa}_i = \kappa_i + N - i \text{ and } y_i = \tilde{\kappa}_{N-i}.$$

Observe that, under this bijection if,

$$\nu = (\nu_1 \geq \cdots \geq \nu_{N+1}) \mapsto x = (x_1 < x_2 < \cdots < x_{N+1}),$$
$$\kappa = (\kappa_1 \geq \cdots \geq \kappa_N) \mapsto y = (y_1 < y_2 < \cdots < y_N),$$

then, $\nu_1 \geq \kappa_1 \geq \nu_2 \geq \cdots \geq \kappa_N \geq \nu_{N+1}$ if and only if $y \in W^{N,N+1}(x)$. Hence, observe that a path of length N is given by a Gelfand-Tsetlin pattern (of type-A) of depth N. See Figure 5.15 for an example.

The Gelfand-Tsetlin graph has a representation theoretic origin, vertices at level N parametrize the irreducible characters of $\mathbb{U}(N)$, the N-dimensional unitary group. The edges correspond to how an irreducible representation of $\mathbb{U}(N)$ when restricted to $\mathbb{U}(N-1)$ splits into irreducibles (since when restricted it becomes reducible).

It is a remarkable Theorem, originally due to Edrei [66] (in an equivalent form) and Voiculescu [161] (see also [160], [117], [27]) that the boundary of the Gelfand-Tsetlin graph can be described explicitly. In order to do this, we need some more definitions.

Let $\mathbb{R}_+^\infty$ denote the product of countably many copies of $\mathbb{R}_+$ and also write $\mathbb{R}_+^{4\infty+2} = \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+ \times \mathbb{R}_+$, equipped with the product topology. Then, consider $\Omega \subset \mathbb{R}_+^{4\infty+2}$ the set of sextuples,

$$\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-),$$

so that,

$$\alpha^\pm = (\alpha_1^\pm \geq \alpha_2^\pm \geq \cdots \geq 0) \in \mathbb{R}_+^\infty \text{ and } \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \cdots \geq 0) \in \mathbb{R}_+^\infty,$$
$$\sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm) \leq \delta^\pm \text{ and } \beta_1^+ + \beta_1^- \leq 1.$$

Note that, Ω is locally compact under the induced topology. Then set,

$$\gamma^\pm = \delta^\pm - \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm)$$

and observe that $\gamma^\pm \geq 0$ and define for $u \in \mathbb{C}^*$ and $\omega \in \Omega$ the function $\Phi(\omega; u)$ given by,

$$\Phi(\omega; u) = e^{\gamma^+(u-1)+\gamma^-(u^{-1}-1)} \prod_{i=1}^\infty \frac{1+\beta_i^+(u-1)}{1-\alpha_i^+(u-1)} \frac{1+\beta_1^-(u^{-1}-1)}{1-\alpha_i^-(u^{-1}-1)}.$$

As its poles do not accumulate to 1, the function $\Phi(\omega; u)$ is holomorphic in a neighbourhood of the unit circle $\mathbb{T} = \{u \in \mathbb{C} : |u| = 1\}$. For $n \in \mathbb{Z}$, we denote its Laurent coefficient by,

$$\phi_n(\omega) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathbb{T}} \Phi(\omega; u) \frac{du}{u^{n+1}}$$

and for a signature $\nu = (\nu_1, \cdots, \nu_N)$ of length N define,

$$\phi_\nu(\omega) = \det\left(\phi_{\nu_i - i + j}(\omega)\right)_{i,j=1}^N$$

and the Markov kernels $\Lambda_N^\infty : \Omega \to V_N$ by,

$$\Lambda_N^\infty(\omega, \nu) = \mathsf{dim}_N(\nu) \phi_\nu(\omega), \forall N \geq 1, \omega \in \Omega, \nu = (\nu_1, \cdots, \nu_N),$$

where $\mathsf{dim}_N(\nu) = \prod_{1 \leq i < j \leq N} \frac{\nu_i - \nu_j + j - i}{j - i}$ is the dimension of a level-N signature $\nu = (\nu_1, \cdots, \nu_N)$.

Then, Ω is the *Feller boundary* of the Gelfand-Tsetlin graph with link from Ω to level N given by Λ_N^∞ (for the Feller property in particular, see Corollary 2.11 of [27]).

BC-type branching graph This graph has a representation theoretic origin as well. For certain values of its multiplicities it describes the branching of the irreducible characters of the Lie groups $\{SO(2N+1)\}_{N\ge1}$, $\{Sp(2N)\}_{N\ge1}$ and $\{O(2N)\}_{N\ge1}$. Vertices at level N are now given by *positive* signatures of length N, namely $\kappa = (\kappa_1 \ge \cdots \ge \kappa_N \ge 0)$ with two vertices $\kappa = (\kappa_1 \ge \cdots \ge \kappa_N \ge 0)$ and $\nu = (\nu_1 \ge \cdots \ge \nu_{N+1} \ge 0)$ being connected by an edge and we write $\kappa \prec_{BC} \nu$, if and only if there exists an "intermediate" signature $\rho = (\rho_1 \ge \cdots \ge \rho_N \ge 0)$ such that,

$$\rho_1 \ge \kappa_1 \ge \cdots \ge \rho_N \ge \kappa_N \text{ and } \nu_1 \ge \rho_1 \ge \cdots \ge \rho_N \ge \nu_{N+1},$$

or equivalently in our notation, under the transformation described previously in the context of the Gelfand-Tsetlin graph $\kappa \mapsto y$, $\rho \mapsto z$ and $\nu \mapsto x$,

$$y \in W^{N,N}(z) \text{ and } z \in W^{N,N+1}(x).$$

The multiplicities are now given in terms of certain coefficients associated to the multivariate $\theta = 1$ Jacobi polynomials, so they depend on two real parameters a, b; see Section 3 of [50] for more details. It is a theorem, originally of Okounkov and Olshanski [118], but also see Section 3 of [50] for a nice exposition and a proof of the Feller property, that the boundary of the BC-type branching graph can be parametrized by the space Ω_{BC} (which *does not* depend on a, b) being the closed subspace of $\mathbb{R}_+^{2\infty+1}$ consisting of points $\omega_{BC} = (\alpha^{BC}, \beta^{BC}, \delta^{BC})$ such that,

$$\alpha^{BC} = (\alpha_1^{BC} \ge \alpha_2^{BC} \ge \cdots \ge 0) \in \mathbb{R}_+^{\infty},\ \beta^{BC} = (1 \ge \beta_1^{BC} \ge \beta_2^{BC} \ge \cdots \ge 0) \in \mathbb{R}_+^{\infty} \text{ and } \sum_{i=1}^{\infty}(\alpha_i^{BC} + \beta_i^{BC}) \le \delta^{BC}.$$

Alternating construction and generalized BC-type branching graph This corresponds to the construction of a general random growth process with a wall in later sections, which we call the *alternating construction*. The graph consists of the vertices and edges of the BC-type branching graph described above, but with more general multiplicities (in particular the BC-type graph is a special case). Of course, these multiplicities are not arbitrary but arise from the *consistent dynamics* between Karlin-McGregor semigroups namely (5.24) and (5.25), or from the branching rules for multivariate Karlin-McGregor polynomials. These polynomials arise as follows: to any family $\{Q_i\}_{i\ge1}$ of orthogonal polynomials in $[0,\infty)$ we can associate a multivariate determinantal version, indexed by $\nu \in W^N$, by $\det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^N / \det\left(x_j^{i-1}\right)_{i,j=1}^N$. Then using the branching rules for these polynomials, see Section 5.7 (also the Appendix) one can obtain the following general multiplicities. In the notation of this chapter, if we define the following (positive) *weight functions* by,

$$(z,y) \in W^{N,N}(\mathbb{N}) \ , \ w_{N,N}(z,y) = \prod_{i=1}^{N} \pi(y_i),$$
$$(x,z) \in W^{N,N+1}(\mathbb{N}) \ , \ w_{N,N+1}(x,z) = \prod_{i=1}^{N} \hat{\pi}(z_i),$$

then, the multiplicities are given by,

$$\mathsf{mult}(x,y) = \sum_{z: y \in W^{N,N}(z), z \in W^{N,N+1}(x)} w_{N,N}(z,y) w_{N,N+1}(x,z).$$

Moreover, observe that for $x \in W^{N+1}$, its dimension in the branching graph is given by the harmonic function from (5.26),

$$\mathsf{dim}_{N+1}(x) = h_{N,N+1}(x) = (\Lambda_{N,N+1}\Lambda_{N,N} \cdots \Lambda_{1,1}\mathbf{1})(x).$$

Under a certain *positive definiteness* assumption, which admittedly can be non-trivial to check (see Appendix), our results from sections 5.8 and 5.9 *partially* describe the boundary of these graphs. More precisely, we first introduce a large class of coherent measures for this graph in Section 5.8. Combining Lemma 5.52 (see also subsection 5.9.2) and the results of subsection 5.11.3 in the Appendix (under this positive definiteness assumption, see Remark 5.75) we show that these coherent sequences are actually extremal.

Remark 5.33. *The projective chains associated to all these graphs can also be recast in terms of branching coefficients of certain families of (symmetric) functions (see Appendix).*

5.5 Examples of consistent dynamics

Before giving any examples we first record some useful facts and fix notation. Throughout this section we will denote the Vandermonde determinant by,

$$\Delta_n(x) = \prod_{1 \le i < j \le n} (x_j - x_i), \ x \in W^n(I).$$

We will consider a difference operator L that is the generator of a birth and death chain or a bilateral birth and death chain with quadratic rates, i.e. so that with $x \in I$,

$$\mathsf{L} = (ax^2 + bx + c)\nabla + (ax^2 + \bar{b}x + \bar{c})\bar{\nabla}.$$

We assume throughout that, $a, b, c, \bar{b}, \bar{c}$ are such that the rates are positive namely,

$$\lambda(x) = (ax^2 + bx + c) > 0 \text{ and } \mu(x) = (ax^2 + \bar{b}x + \bar{c}) > 0 \ , \ \forall x \in I$$

and that conditions (5.8),(5.9) or (5.10),(5.11),(5.12) and (5.13) respectively are always satisfied for all chains considered in this subsection. Finally, observe that we need the leading coefficient a to be the same for both rates.

Now, with all these requirements in place a direct computation (see e.g. [60] Proposition 6.2.1) gives that,

$$\sum_{i=1}^{n} \mathsf{L}_{x_i}\Delta_n(x) = \left(a\frac{n(n-1)(n-2)}{3} + (b-\bar{b})\frac{n(n-1)}{2}\right)\Delta_n(x),\ x \in W^n(I),$$

where each L_{x_i} is a copy of the difference operator L acting in the x_i variable. So that, we can h-transform n independent copies of L-chains by Δ_n to stay in $W^n(I)$.

Define the following operator from functions on $W^n(I)$ to functions on $W^{n+1}(I)$, these when viewed as Markov kernels from $W^{n+1}(I)$ to $W^n(I)$ are the links that appear in the Gelfand-Tsetlin graph by,

$$(\mathfrak{L}^{\mathrm{Vnd}}_{n\to n+1}f)(x) = \frac{n!}{\Delta_{n+1}(x)}\sum_{y\in W^{n,n+1}(x)} \Delta_n(y)f(y),\ x \in W^n(I).$$

Then, we have the following lemma.

Lemma 5.34. *For $n \geq 1$, the kernels $\mathfrak{L}^{\mathrm{Vnd}}_{n\to n+1}$ are Feller.*

Proof. In order to prove this, it suffices to apply the kernel $\mathfrak{L}^{\mathrm{Vnd}}_{n\to n+1}$ to a delta function δ_y and show that $(\mathfrak{L}^{Vnd}_{n\to n+1}\delta_y)(x)$ vanishes as $x \to \infty$. This can be readily checked, see e.g. Proposition 3.3 of [28] for the details. ◻

Now, suppose that we are given as above the following birth and death (reflecting at the origin, $\mu(0)=0$) or bilateral ($I=\mathbb{Z}$) chain with generator $\mathcal{D}=\mathsf{L}$ so that,

$$\mathcal{D}(x,y) = \begin{cases} ax^2+bx+c & y=x+1 \\ -(ax^2+bx+c)-(ax^2+\bar{b}x+\bar{c}) & y=x \\ ax^2+\bar{b}x+\bar{c} & y=x-1 \end{cases}.$$

Then, a simple computation gives us that the h-transform of the chain with generator $\mathcal{D}$ by the strictly positive function $\hat{\pi}^{-1}$ (which is an eigenfunction with eigenvalue $b-\bar{b}$) is the (reflecting) birth and death (or bilateral birth and death chain) with generator $\tilde{\mathcal{D}}$ with rates,

$$\tilde{\mathcal{D}}(x,y) = \begin{cases} a(x+1)^2+b(x+1)+c & y=x+1 \\ -(a(x+1)^2+b(x+1)+c)-(ax^2+\bar{b}x+\bar{c}) & y=x \\ ax^2+\bar{b}x+\bar{c} & y=x-1 \end{cases}.$$

Moreover, we define $\left(P^{\Delta_{n+1}}_{n+1}(t);t\geq 0\right)$ to be the Karlin-McGregor semigroup of $n+1$ copies of $\mathcal{D}$-chains h-transformed by Δ_{n+1} and similarly $\left(\tilde{P}^{\Delta_n}_n(t);t\geq 0\right)$ to be the Karlin-

McGregor semigroup of n copies of $\tilde{\mathcal{D}}$-chains h-transformed by Δ_n. Then as expected these possess the Feller property.

Lemma 5.35. *The semigroups $\left(P_{n+1}^{\Delta_{n+1}}(t); t \geq 0\right)$ and $\left(\tilde{P}_n^{\Delta_n}(t); t \geq 0\right)$ are Feller for any n.*

Proof. This, again easily follows by applying these semigroups to δ_y and making use of the fact that the one dimensional transition densities in the Karlin-McGregor semigroups satisfy $p_t(x_i, y_j), \tilde{p}_t(x_i, y_j) \to 0$ as $x_i \to \infty$ (or $-\infty$) and that moreover $\Delta_n(x) \geq 1$. $\square$

Then, Theorem 5.18 and in particular, the intertwining relation (5.29) immediately gives the following proposition which is the main result of this subsection.

Proposition 5.36. $P_{n+1}^{\Delta_{n+1}}(t)\mathfrak{L}_{n\to n+1}^{\mathrm{Vnd}} f = \mathfrak{L}_{n\to n+1}^{\mathrm{Vnd}} \tilde{P}_n^{\Delta_n}(t) f$*, for $n \geq 1$, $f \in C_0(W^n(I))$ and $t \geq 0$.*

We now, list several interesting applications of this proposition. For $a = b = \bar{b} = 0$ and $c, \bar{c} > 0$, we obtain the well known intertwining between non-colliding (asymmetric) continuous time random walks.

For a linear birth and death chain, i.e. with a parameter $\theta > 0$ and rates given by,

$$\mathcal{D}_\theta(x,y) = \begin{cases} x+\theta & y = x+1 \\ -2x-\theta & y = x \\ x & y = x-1 \end{cases},$$

we get that,

$$\tilde{\mathcal{D}}_\theta(x,y) = \begin{cases} x+\theta+1 & y = x+1 \\ -2x-\theta-1 & y = x \\ x & y = x-1 \end{cases}.$$

Observe that $\tilde{\mathcal{D}}_\theta = \mathcal{D}_{\theta+1}$, the birth rate or equivalently the drift to the right of the preceding level increased by 1, in particular such a construction cannot be iterated indefinitely. Moreover, Proposition 5.36 gives the discrete analogue of the intertwining between $n+1$ non-intersecting squared Bessel processes of dimension d abbreviated by BESQ(d) and n non-intersecting BESQ($d+2$) (see Subsection 1.3.2 of Chapter 1).

We can also consider the Meixner process, which is the analogue of the Laguerre diffusion (a BESQ process with a restoring drift towards the origin, for certain choices of parameters the modulus of Ornstein Uhlenbeck processes, see Subsection 1.3.2) with parameters $r, \theta > 0$,

$$\mathcal{D}_{r,\theta}^{\mathrm{Me}}(x,y) = \begin{cases} r(x+\theta) & y = x+1 \\ -r(x+\theta)-(r+1)x & y = x \\ (r+1)x & y = x-1 \end{cases},$$

then,

$$\tilde{\mathcal{D}}^{\mathrm{Me}}_{r,\theta}(x,y)=\begin{cases} r(x+\theta+1) & y=x+1 \\ -r(x+\theta+1)-(r+1)x & y=x \\ (r+1)x & y=x-1 \end{cases}.$$

Similarly as above, we see that $\tilde{\mathcal{D}}^{\mathrm{Me}}_{r,\theta}=\mathcal{D}^{\mathrm{Me}}_{r,\theta+1}$, so that the drift to the right has decreased from the preceding level, or when thinking in terms of the couplings, the birth rate for the autonomous particles is greater by 1.

As a final example, we consider the bilateral birth and death chain studied by Borodin and Olshanski in [28] (see also Section 3.6 of Chapter 3), with $u,u',v,v'\in\mathbb{C}$ satisfying the assumptions in section 5.1 therein (these ensure well-posedness and non-explosion, moreover note that although the parameters can be complex, they really correspond to 4 free real parameters),

$$\mathcal{D}^{\mathbb{U}(\infty)}_{u,u',v,v'}(x,y)=\begin{cases} (x-u)(x-u') & y=x+1 \\ -(x-u)(x-u')-(x+v)(x+v') & y=x \\ (x+v)(x+v') & y=x-1 \end{cases},$$

so that,

$$\tilde{\mathcal{D}}^{\mathbb{U}(\infty)}_{u,u',v,v'}(x,y)=\begin{cases} (x+1-u)(x+1-u') & y=x+1 \\ -(x+1-u)(x+1-u')-(x+v)(x+v') & y=x \\ (x+v)(x+v') & y=x-1 \end{cases}.$$

As before note the following fact, $\tilde{\mathcal{D}}^{\mathbb{U}(\infty)}_{u,u',v,v'}=\mathcal{D}^{\mathbb{U}(\infty)}_{u-1,u'-1,v,v'}$. Then, Proposition 5.36 above immediately gives as a corollary Theorem 6.1 of [28]. This along with the *method of intertwiners* (see Subsection 5.4.2), constructs a Feller process on the boundary Ω of the Gelfand-Tsetlin graph. We note that the motivation behind these specific rates stems from the fact that the corresponding semigroups leave invariant the so called *zw*-measures, which are consistent measures on the Gelfand-Tsetlin graph and whose decomposition into extremal coherent measures is the *problem of harmonic analysis* on the infinite dimensional unitary group $\mathbb{U}(\infty)$ (for more details see [120]).

Characterization of Vandermonde intertwiners for push-block dynamics The choice of quadratic rates might have seemed a bit arbitrary. We now proceed to briefly explain its significance. More specifically, we show that in order for the Vandermonde links,

$$(\mathfrak{L}^{\mathrm{Vnd}}_{n\to n+1}f)(x)=\frac{n!}{\Delta_{n+1}(x)}\sum_{y\in W^{n,n+1}(x)}\Delta_n(y)f(y),\ x\in W^n(I),$$

to intertwine the levels of the (type-A) Gelfand-Tsetlin pattern valued process moving according to the push-block dynamics considered in the two-level couplings of this chapter (or c.f. equality (5.29), for the semigroups for each level to be consistent with these links) then, the rates $\lambda(x)$ and $\mu(x)$ must be quadratic functions of $x \in I$, with coefficients related as shown below in displays (5.56) and (5.57).

Starting from the process of the two first levels, taking values in $W^{1,2}$, it is easy to see from relation (5.28) that we need $\hat{\pi}^{-1}$ to be an eigenfunction of the generator $\hat{\mathcal{D}}$ for the resulting intertwining kernel to be given by,

$$\frac{1}{x_2 - x_1}\mathbf{1}(x_1 \leq y < x_2).$$

Since $\hat{\mathcal{D}}$ is reversible with respect to $\hat{\pi}$, this requirement is equivalent to the fact that the transpose (when viewed as an infinite matrix indexed by $\mathbb{N}$ or $\mathbb{Z}$) of $\hat{\mathcal{D}}$ minus some constant times the identity matrix ($\hat{\mathcal{D}}^{\mathrm{T}} - \mathrm{const} \times Id$) is the generator of a birth and death (or bilateral) chain with rates,

$$\tilde{\mathcal{D}}(x,y) = \begin{cases} \tilde{\lambda}(x) = \lambda(x+1) & y = x+1 \\ -\lambda(x+1) - \mu(x) & y = x \\ \tilde{\mu}(x) = \mu(x) & y = x-1 \end{cases}.$$

Now this is true, if and only if, for some constant c_0,

$$\lambda(x+1) + \mu(x) - \mu(x+1) - \lambda(x) = c_0 \, , \forall x \in \mathbb{Z}.$$

Then, moving to the two-level process taking values in $W^{2,3}$, an analogous consideration (with λ, μ still denoting the birth and death rates of the chains on the 2^{nd} level) leads to the extra requirement that,

$$\lambda(x+2) + \mu(x) - \mu(x+1) - \lambda(x+1) = c_1 \, , \forall x \in \mathbb{Z}.$$

These two conditions are now sufficient to characterize $\lambda(x)$ and $\mu(x)$ as quadratic functions of x. Let $\Lambda(x) = (\nabla\lambda)(x)$ and $M(x) = (\nabla\mu)(x)$ so that,

$$\Lambda(x) - M(x) = c_0,$$
$$\Lambda(x+1) - M(x) = c_1.$$

Observe that, with $n \geq 0$ we have $\Lambda(x+n) - M(x) = \Lambda(x+n) - \Lambda(x+n-1) + \Lambda(x+n-1) - M(x) = c_1 - c_0 + \Lambda(x+n-1) - M(x) = \cdots = n(c_1 - c_0) + c_0$ and similarly for n negative. Thus,

$$\Lambda(y) = y(c_1 - c_0) + c_0 + M(0),$$
$$M(y) = y(c_1 - c_0) + M(0).$$

From these, we obtain,

$$\mu(y) = \frac{y(y-1)}{2}(c_1 - c_0) + (\mu(1) - \mu(0))y + \mu(0), \tag{5.56}$$

$$\lambda(y) = \frac{y(y-1)}{2}(c_1 - c_0) + (c_0 + \mu(1) - \mu(0))y + \lambda(0), \tag{5.57}$$

where $\lambda(1) = c_0 + \mu(1) - \mu(0) + \lambda(0)$ so that $c_0 = \mu(1) - \mu(0) + \lambda(0) - \lambda(1)$ and $\lambda(2) = c_1 + \lambda(0) + \mu(1) + \mu(0)$ so that $c_1 = \lambda(2) - \lambda(0) - \mu(1) - \mu(0)$.

In conclusion, at an *algebraic level* we need to specify five positive real parameters $\lambda(0), \lambda(1), \lambda(2), \mu(0), \mu(1)$. Of course in addition to that, we need $\mu(y), \lambda(y) > 0$ and that the well-posedness conditions (5.8), (5.9) or (5.10), (5.11), (5.12) and (5.13) respectively are satisfied. Finally, if we denote by $\mathsf{r}_1^+(x), \mathsf{r}_1^-(x)$ the quadratic birth and death rates respectively of the single chain at level 1 then, the rates for the chains at level n are given by $\mathsf{r}_n^+(x) = \mathsf{r}_1^+(x+n-1)$ and $\mathsf{r}_n^-(x) = \mathsf{r}_1^-(x)$.

Intertwining relations for dynamics on BC-type graphs The aim of this subsection is to prove Proposition 5.39 below, first proven as Theorem 5.1 in [50] by Cuenca. We will use the following notation. In all that follows, $I = \mathbb{N}$ and we define,

$$W_{\mathrm{BC}}^{n,n+1} = \{(x,y) \in (W^{n+1}, W^n) : \exists\, z \in W^n, \text{ such that } y \in W^{n,n}(z), z \in W^{n,n+1}(x)\}.$$

Analogously to $W^{n,n+1}$ we define $W_{\mathrm{BC}}^{n,n+1}(x)$ for $x \in W^{n+1}$.

Moreover, we consider the following rates for a $\mathcal{D}$-chain depending on 4 parameters (u, u', a, b), which satisfy the relations (5.1) in [50] (these conditions ensure positivity of the rates and non-explosivity of the chain and will not be recalled since they don't affect the essentially algebraic arguments below), with $\beta_{u,u'}$ denoting the *birth rate* and $\delta_{u,u'}$ the *death rate*, for $x \in \mathbb{N}$,

$$\beta_{u,u'}(x) = \frac{(x+a+b+1)(x+a+1)(x-u)(x-u')}{(2x+a+b+1)(2x+a+b+2)},$$

$$\delta_{u,u'}(x) = \frac{x(x+b)(x+u+a+b+1)(x+u'+a+b+1)}{(2x+a+b+1)(2x+a+b)}.$$

The parameters (a,b) will be fixed throughout so we suppress any dependence of $\beta_{u,u'}$ and $\delta_{u,u'}$ on them. Now, define the following functions $\mathsf{f}, \mathsf{g}, \mathsf{B}$ again depending on (a,b) but *not* on u and u' by,

$$\mathsf{f}(x) = \frac{(2x+a+b+2)x!\Gamma(x+b+1)}{\Gamma(x+a+b+2)\Gamma(x+a+2)}, \ x \in \mathbb{N},$$

$$\mathsf{g}(y) = \frac{(2y+a+b+1)\Gamma(y+a+b+1)\Gamma(y+a+1)}{y!\Gamma(y+b+1)}, \ y \in \mathbb{N},$$

$$\mathsf{B}(x,y) = \frac{1}{2}\mathsf{f}(x)\mathsf{g}(y), \ x, y \in \mathbb{N}.$$

Define the function F_n on W^n by,

$$\mathsf{F}_n(x) = \prod_{i<j}^{n} \left(\left(x_j + \frac{a+b+1}{2} \right)^2 - \left(x_i + \frac{a+b+1}{2} \right)^2 \right).$$

Furthermore, define the following kernel,

$$(\mathfrak{L}^{\mathbf{BC}}_{n\to n+1} f)(x) = \frac{2^n n! \Gamma(n+a+1)}{\Gamma(a+1)\mathsf{F}_{n+1}(x)} \sum_{y \in W^{n,n+1}_{\mathrm{BC}}(x)} \mathsf{F}_n(y) f(y) \sum_{z: y \in W^{n,n}(z), z \in W^{n,n+1}(x)} \prod_{i=1}^{n} \mathsf{B}(z_i, y_i), \ x \in W^{n+1}.$$

Then, we have the following lemma originally proven in [50].

Lemma 5.37. *For $n \geq 1$, the kernels $\mathfrak{L}^{\boldsymbol{BC}}_{n\to n+1}$ are Feller.*

Proof. The fact that these are Markov, i.e. correctly normalized, comes from the branching of the normalized Jacobi polynomials, see Section 3 of [50]. Moreover, to show that they are Feller, it again suffices to check it for a delta function; however the situation is a bit more involved than for $\mathfrak{L}^{Vnd}_{n\to n+1}$, see Proposition 3.1 of [50] for the details. $\square$

Denote by $\left(P^{u,u'}_n(t); t \geq 0\right)$ the Karlin-McGregor semigroup associated to n $\mathcal{D}$-chains with birth and death rates $\beta_{u,u'}$ and $\delta_{u,u'}$ respectively. It can be checked, see Lemma 4.12 of [50], that F_n is a positive eigenfunction of $P^{u,u'}_n(t)$ with eigenvalue $e^{c_n t}$, where $c_n = \frac{n(n-1)(n-2)}{3} - \frac{n(n-1)}{2}(u+u'+b)$ (this fact can also be obtained via iteration of the results below) so that in particular, we can define the honest Markov semigroup $\left(P^{u,u',\mathsf{F}_n}_n(t); t \geq 0\right)$ given by the h-transform of $\left(P^{u,u'}_n(t); t \geq 0\right)$ by F_n. Then, under the assumptions on (u, u', a, b) referred to above we have:

Lemma 5.38. *For $n \geq 1$, the semigroups $\left(P^{u,u',\mathsf{F}_n}_n(t); t \geq 0\right)$ are Feller.*

Proof. This as before, immediately follows from the fact that the one dimensional transition densities that go in the Karlin-McGregor semigroups are Feller along with the fact that $F_n(x) \geq 1$. $\square$

Finally, the following proposition along with the method of intertwiners immediately gives a Feller process on the boundary Ω_{BC} of the type-BC branching graph.

Proposition 5.39. *$P^{u+1,u'+1,\mathsf{F}_{n+1}}_{n+1}(t)\mathfrak{L}^{\boldsymbol{BC}}_{n\to n+1} f = \mathfrak{L}^{\boldsymbol{BC}}_{n\to n+1} P^{u,u',\mathsf{F}_n}_n(t) f$, for $n \geq 1$, $f \in C_0(W^n)$, $t \geq 0$.*

Again, the interest in these specific rates stems from the fact that they preserve the so called z-measures, which are the analogues of the zw-measures mentioned previously, for the problem of harmonic analysis on infinite dimensional BC-type groups. For more details and a complete study of the z-measures see the recent paper [50].

Proposition 5.39 will follow from the two relations given in Proposition 5.40 below, which reveal a "hidden" dynamic on "intermediate signatures" (see Okounkov's paper [114] and the references therein for more about these). In fact, this is exactly the dynamic followed by the projection on the even levels ($x^{(i,i)}$ in our notation), if one constructs a symplectic Gelfand-Tsetlin pattern valued process, that links (on odd levels) the semigroups $\left(P_{n+1}^{u+1,u'+1,\mathsf{F}_{n+1}}(t); t \geq 0\right)$ and $\left(P_{n}^{u,u',\mathsf{F}_n}(t); t \geq 0\right)$ and initializes it according to a Gibbs measure (see Proposition 5.30).

Some more definitions are necessary. Let the functions $\hat{\mathsf{F}}_n$ and $\bar{\mathsf{F}}_{n+1}$ on W^n and W^{n+1} respectively be given by,

$$\hat{\mathsf{F}}_n(z) = \sum_{y \in W^{n,n}(z)} \prod_{i=1}^{n} \mathsf{g}(y_i)\mathsf{F}_n(y),\ z \in W^n,$$
$$\bar{\mathsf{F}}_{n+1}(x) = \sum_{z \in W^{n,n+1}(x)} \prod_{i=1}^{n} \mathsf{f}(z_i)\hat{\mathsf{F}}_n(z),\ z \in W^{n+1}.$$

Moreover, we define the following Markov kernels $\mathfrak{L}_{n,n}^{\mathbf{BC}}$ from W^n to W^n, and $\mathfrak{L}_{n,n+1}^{\mathbf{BC}}$ from W^{n+1} to W^n respectively by,

$$(\mathfrak{L}_{n,n}^{\mathbf{BC}} f)(z) = \frac{1}{\hat{\mathsf{F}}_n(z)} \sum_{y \in W^{n,n}(z)} f(y) \prod_{i=1}^{n} \mathsf{g}(y_i)\mathsf{F}_n(y),\ z \in W^n,$$
$$(\mathfrak{L}_{n,n+1}^{\mathbf{BC}} f)(x) = \frac{1}{\bar{\mathsf{F}}_{n+1}(x)} \sum_{z \in W^{n,n+1}(x)} f(z) \prod_{i=1}^{n} \mathsf{f}(z_i)\hat{\mathsf{F}}_n(z),\ x \in W^{n+1}.$$

Observe that, we have the composition property,

$$\mathfrak{L}_{n\to n+1}^{\mathbf{BC}} = \mathfrak{L}_{n,n+1}^{\mathbf{BC}} \circ \mathfrak{L}_{n,n}^{\mathbf{BC}}$$

and from comparing the two expressions in order to get the right normalization constant, we have,

$$\bar{\mathsf{F}}_{n+1}(x) = \frac{\Gamma(a+1)}{n!\Gamma(n+a+1)}\mathsf{F}_{n+1}(x),\ x \in W^{n+1}.$$

Finally, we denote by $\left(\mathsf{P}_n^{u,u',\hat{\mathsf{F}}_n}(t); t \geq 0\right)$ the Karlin-McGregor semigroup associated with n birth and death chains with *birth rate*,

$$\frac{\mathsf{g}(x)\beta_{u,u'}(x)}{\mathsf{g}(x+1)},\ x \in \mathbb{N},$$

and *death rate*,

$$\frac{\mathsf{g}(x+1)\delta_{u,u'}(x+1)}{\mathsf{g}(x)},\ x \in \mathbb{N},$$

that is moreover Doob's h-transformed by $\hat{\mathsf{F}}_n$. The fact that, this is indeed an eigenfunction of n copies of such birth and death chains follows (recursively) from relation (5.58) of Proposition 5.40 below. This semigroup, $\left(\mathsf{P}_n^{u,u',\hat{\mathsf{F}}_n}(t); t \geq 0\right)$ that is driving the evolution of n

non-intersecting birth and death chains is the "hidden" dynamic alluded to above. Now, Proposition 5.39 is an immediate consequence of the following result.

Proposition 5.40. *For $n \geq 1$ and $t \geq 0$, we have the intertwining relations:*

$$\mathsf{P}_n^{u,u',\hat{\mathsf{F}}_n}(t)\mathfrak{L}_{n,n}^{BC} = \mathfrak{L}_{n,n}^{BC} P_n^{u,u',\mathsf{F}_n}(t) \tag{5.58}$$

$$P_{n+1}^{u+1,u'+1,\mathsf{F}_{n+1}}(t)\mathfrak{L}_{n,n+1}^{BC} = \mathfrak{L}_{n,n+1}^{BC}\mathsf{P}_n^{u,u',\hat{\mathsf{F}}_n}(t) \tag{5.59}$$

Proof. In the setting of Theorem 5.20, with n and n particles on each of the X and Y levels, we choose the $\mathcal{D}$-chains (the Y-level) to have rates given by,

$$\lambda(x) = \frac{\mathsf{g}(x+1)\delta_{u,u'}(x+1)}{\mathsf{g}(x)},\ x \in \mathbb{N},$$

$$\mu(x) = \frac{\mathsf{g}(x-1)\beta_{u,u'}(x-1)}{\mathsf{g}(x)},\ x \in \mathbb{N}.$$

Observe that, by performing an h-transform by the function $\prod_{i=1}^n \pi^{-1}(y_i)\mathsf{g}(y_i)\mathsf{F}_n(y)$ the evolution of these chains is driven by $\left(P_n^{u,u',\mathsf{F}_n}(t); t \geq 0\right)$ and thus we obtain (5.58).

Now, in the setting of Theorem 5.18 with n and $n+1$ particles, let the $\mathcal{D}$-chains (the X-level in this new setting, note that these are different from the ones considered above) have birth rate given by $\beta_{u+1,u'+1}(x)$ and death rate given by $\delta_{u+1,u'+1}(x)$. Then, performing an h-transform of the corresponding n $\hat{\mathcal{D}}$-chains (the Y-level) by the function $\prod_{i=1}^n \hat{\pi}^{-1}(z_i)\mathsf{f}(z_i)\hat{\mathsf{F}}_n(z)$ we obtain (5.59) after we observe the following compatibility relations between the jump rates,

$$\beta_{u+1,u'+1}(x+1)\frac{\mathsf{f}(x+1)}{\mathsf{f}(x)} = \mu(x+1) = \beta_{u,u'}(x)\frac{\mathsf{g}(x)}{\mathsf{g}(x+1)},\ x \in \mathbb{N}, \tag{5.60}$$

$$\delta_{u+1,u'+1}(x)\frac{\mathsf{f}(x-1)}{\mathsf{f}(x)} = \lambda(x) = \delta_{u,u'}(x+1)\frac{\mathsf{g}(x+1)}{\mathsf{g}(x)},\ x \in \mathbb{N}. \tag{5.61}$$

To see that these relations hold, first note that by making use of $\Gamma(x+1) = x\Gamma(x)$ we obtain the following, for ratios of f and g at consecutive points,

$$\frac{\mathsf{f}(x+1)}{\mathsf{f}(x)} = \frac{(2x+a+b+4)(x+1)(x+b+1)}{(2x+a+b+2)(x+a+b+2)(x+a+2)},\ x \in \mathbb{N},$$

$$\frac{\mathsf{g}(x+1)}{\mathsf{g}(x)} = \frac{(2x+a+b+4)(x+1)(x+b+1)}{(2x+a+b+2)(x+a+b+2)(x+a+2)},\ x \in \mathbb{N}.$$

Similarly, we have relations for ratios of the birth and death rates with different parameters,

$$\frac{\beta_{u+1,u'+1}(x+1)}{\beta_{u,u'}(x)} = \frac{(x+a+b+2)(x+a+2)(2x+a+b+1)(2x+a+b+2)}{(2x+a+b+3)(2x+a+b+4)(x+a+b+1)(x+a+1)},\ x \in \mathbb{N},$$

$$\frac{\delta_{u+1,u'+1}(x)}{\delta_{u,u'}(x+1)} = \frac{x(x+b)(2x+a+b+3)(2x+a+b+2)}{(2x+a+b+1)(2x+a+b)(x+1)(x+1+b)},\ x \in \mathbb{N}.$$

Using these, (5.60) and (5.61) can be readily checked and we are done. □

Strong Stationary Duals Here, we briefly point out the close connection to the theory of Strong Stationary Duality. The setup is that of $W^{1,1}$ and with $I = \mathbb{N}$ i.e. X and Y each consist of a single particle. We define the cumulative of π, by $\sum_{0\le y\le x} \pi(y)$. Thus, Theorem 5.20 gives that if a $\hat{\mathcal{D}}$-chain (X-level) is being kept above a (reflecting) $\mathcal{D}$-chain (Y-level) via the push-block mechanism we have been studying; then if the $\mathcal{D}$-chain is distributed initially according to $\frac{\pi(y)}{\sum_{0\le y\le x}\pi(y)} 1(y \le x)$, the evolution of the projection on the X-particle is that of a $\hat{\mathcal{D}}$-chain h-transformed by $\sum_{0\le y\le x} \pi(y)$ (see for example Theorem 5.5 of [57] in the discrete time case).

Remark 5.41. *Using the results of this chapter, we can also obtain Theorem 2.3 of [166] which studies a process in a symplectic Gelfand-Tsetlin pattern. Similarly, we could consider pure-birth chains, which strictly speaking are not covered by the results of this work, since we assume that we are dealing with positive death rates* $(\mu(x))_{x\in I} > 0$, *but with entirely analogous considerations Theorem 2.1 of [166] can also be recovered by the methods that are presented here.*

5.6 Birth and death chain orthogonal polynomials

We will now recall the well known connection, between the probabilistic world of birth and death chains and the analytic counterpart of their associated orthogonal polynomials on the positive half line. The main references for this subsection will be the seminal papers of Karlin and McGregor, [92] and [93], where most of the theory was laid out. From here onwards, we fix a birth and death chain with generator $\mathcal{D}$, reflecting at 0, with rates $(\lambda(\cdot), \mu(\cdot))$ and symmetrizing measure $\pi(\cdot)$. As usual we shall also denote by $\hat{\mathcal{D}}$ the generator of its Siegmund dual (which is absorbed at -1) with rates $(\hat{\lambda}(\cdot), \hat{\mu}(\cdot))$ and symmetrizing measure $\hat{\pi}(\cdot)$. We will also, often write λ_k for $\lambda(k)$, π_k for $\pi(k)$ and so on.

We begin by defining the following family of polynomials $\{Q_i\}_{i\ge 0}$ by the three term recursion (note that $\mu(0) = 0$),

$$\begin{aligned}
Q_0(x) &= 1,\\
-xQ_0(x) &= -(\lambda(0) + \mu(0))Q_0(x) + \lambda(0)Q_1(x),\\
-xQ_n(x) &= \mu(n)Q_{n-1}(x) - (\lambda(n) + \mu(n))Q_n(x) + \lambda(n)Q_{n+1}(x).
\end{aligned}$$

Then, see Theorem 1 of [93], there exists at least one measure $\mathfrak{w}(dx)$ on $\mathbb{R}_+ = \{0 \le x < \infty\}$, such that these polynomials are orthogonal with respect to $\mathfrak{w}(dx)$, so that,

$$\int_0^\infty Q_i(x)Q_j(x)\mathfrak{w}(dx) = \frac{1}{\pi(j)}\delta_{ij}.$$

For such a *moment problem* to be *determinate*, so that the measure $\mathfrak{w}$ is unique, when $\mu(0) = 0$, as in the case of the $\mathcal{D}$-chain, it suffices for the backwards equation to have a unique solution (see [93], Theorem 14). In particular, any of the conditions in section 5.2 that ensure the well-posedness of the backwards equation are enough for determinacy. In such a case, we have that,

$$\mathfrak{w}(dx) = d\mathfrak{w}(x),$$

where $\mathfrak{w}(x)$ is a real valued non-decreasing function, being continuous on the left, with $\mathfrak{w}(x) = 0$ for $x \leq 0$ and $\mathfrak{w}(\infty) = 1$. We will denote by $\mathfrak{I} = [I^-, I^+] \subset [0, \infty]$ the support, $\mathrm{supp}(\mathfrak{w})$ of the measure $\mathfrak{w}$. These orthogonal polynomials provide the following spectral expansion of the transition density (see [93] for example) that will be useful for us,

$$p_t(i, j) = \pi(j) \int_0^\infty e^{-tx} Q_i(x) Q_j(x) d\mathfrak{w}(x). \tag{5.62}$$

Remark 5.42 (Explicit examples)**.** *We give some simple examples for $\lambda(\cdot), \mu(\cdot)$ such that the corresponding orthogonal polynomials $Q_i(x)$ and spectral measures $\mathfrak{w}(dx)$ are explicit. The following rates were considered by Cerenzia and Kuan in [47], depending on two real parameters $\alpha, \beta > -1$:*

$$\lambda(n) = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} \frac{2(n+\alpha+1)}{2n+\alpha+\beta+2},$$
$$\mu(n) = \frac{n+\beta}{2n+\alpha+\beta} \frac{2n}{2n+\alpha+\beta+1}.$$

They give rise to the Jacobi polynomials $Q_i^{\alpha,\beta}(x)$ orthogonal in $[0, 2]$ with respect to the weight $\mathfrak{w}(dx) = \mathfrak{w}_{\alpha,\beta}(dx)$:

$$\mathfrak{w}_{\alpha,\beta}(dx) = Z(\alpha, \beta) x^\alpha (2-x)^\beta dx,$$

for some normalization constant $Z(\alpha, \beta)$. For $\alpha = \beta = -\frac{1}{2}$ these specialize to the model studied by Borodin and Kuan in [23] related to $O(\infty)$ while for $-\alpha = \beta = \frac{1}{2}$ they specialize to the model studied by Cerenzia [46] related to $Sp(\infty)$. The associated orthogonal polynomials in both cases are the Chebyshev (which are specializations of the Jacobi polynomials).

The following examples are taken from Section 3.1 of [140]. Further explicit examples can be found in the references therein. In all cases $\mathfrak{w}(dx)$ is actually a discrete measure with atoms of mass $w(n)$ at the positive integers $n \in \mathbb{N}$. The associated (2+1)-dimensional growth and decay processes were not studied before.

The so called $M/M/\infty$ queue is a birth and death process with rates and symmetrizing measure given by:

$$\lambda(n) \equiv \lambda, \quad \mu(n) = \mu n, \quad \pi(n) = \left(\frac{\lambda}{\mu}\right)^n / n!.$$

The orthogonal polynomials associated to it are $Q_n(x) = C_n\left(\frac{x}{\mu}; \frac{\lambda}{\mu}\right)$ where $C_n(x; a)$ are the Charlier polynomials defined by:

$$0 = C_{n-1}(x; a) + (x - a - n) C_n(x; a) + a C_{n+1}(x; a),$$

with $C_0(x;a) = 1, C_{-1}(x;a)$. These are orthogonal with respect to the Poisson distribution:

$$w(n) = \frac{a^n e^{-a}}{n!};\ n = 0, 1, \cdots$$

More precisely:

$$\sum_{n=0}^{\infty} C_i\left(n; \frac{\lambda}{\mu}\right) C_j\left(n; \frac{\lambda}{\mu}\right) \frac{(\lambda/\mu)^n}{n!} e^{-(\lambda/\mu)} = \frac{\delta_{ij}}{\pi(j)}.$$

Moreover, the polynomials associated to the birth and death chain with linear rates:

$$\lambda(n) = (n+\beta)\lambda, \quad \mu(n) = n\mu,$$

are the so called Meixner polynomials (see Section 1.3.2 in [140]). Finally for finite birth and death chains one can also obtain the dual Hahn, Krawtchouk and Racah polynomials, see [140].

We also define the polynomials $\{\hat{Q}_i\}_{i\geq 0}$, associated to the dual chain with generator $\hat{\mathcal{D}}$. So that, in the recursion above the rates (λ, μ) are replaced by the dual rates $\left(\hat{\lambda}, \hat{\mu}\right)$. In particular, the new recursion is given by,

$$-x\hat{Q}_n(x) = \lambda(n)\hat{Q}_n(x) - (\mu(n+1) + \lambda(n))\hat{Q}_n(x) + \mu(n+1)\hat{Q}_{n+1}(x).$$

Since now $\hat{\mu}(0) = \lambda(0) > 0$ (recall the $\hat{\mathcal{D}}$-chain gets absorbed at -1), in order for the moment problem to be determinate, we need to further require (see [92] or [93]),

$$\sum_{j=0}^{\infty} \hat{\pi}(j) \left(\sum_{k=0}^{j} \pi(k)\right)^2 = \infty.$$

A sufficient, easier to check in practise, condition for this is (see unnumbered display after equation (0.11) on page 367 of [92]),

$$\sum_{n=1}^{\infty} \frac{1}{\hat{\mu}(n)} = \sum_{n=1}^{\infty} \frac{1}{\lambda(n)} = \infty.$$

In such a case (of determinacy), the dual spectral measure, denoted by $d\hat{\mathfrak{w}}(x)$, satisfies the following key relation (see [92] section 6),

$$d\hat{\mathfrak{w}}(x) = \frac{x d\mathfrak{w}(x)}{\lambda(0)}.$$

So that in particular, the supports are equal $\mathrm{supp}(\hat{\mathfrak{w}}) = \mathrm{supp}(\mathfrak{w}) = \mathfrak{I}$. From now on, we assume that both moment problems are determinate with unique solutions $\mathfrak{w}(\cdot)$ and $\hat{\mathfrak{w}}(\cdot)$ respectively.

We will denote by $\langle \cdot, \cdot \rangle_{\mathsf{m}}$ the L^2 inner product with measure m. By Corollary 2.3.3

of [3] we obtain that, since the solution of the moment problem is unique, the polynomials $\{Q_i\}_{i\geq 0}$ are dense in $L^2(\mathfrak{I}, \mathfrak{w})$. Hence, for $f \in L^2(\mathfrak{I}, w)$,

$$f = \sum_{k=0}^{\infty} \langle Q_k, f\rangle_{\mathfrak{w}} Q_k \pi(k), \tag{5.63}$$

with the series converging in the $L^2(\mathfrak{I}, \mathfrak{w})$ sense. We will furthermore, mainly be interested in functions $f \in L^2$ for which this expansion actually converges uniformly. By Theorem 6 of [93], we have that for $f(x) = Q_i(x)e^{-tx}$ the series,

$$f(x) = \sum_{k=0}^{\infty} \langle Q_k, f\rangle_{\mathfrak{w}} Q_k(x) \pi(k), \tag{5.64}$$

converges absolutely, for $t \geq 0$ and all $x \in \mathbb{C}$, the convergence being uniform over every bounded set, $\left\{(t,x) : 0 \leq t \leq T \text{ and } |x| \leq R\right\}$. Moreover, we have the following bound,

$$\sum_{k=0}^{\infty} |\langle Q_k, f\rangle_{\mathfrak{w}}||Q_k(x)|\pi(k) \leq e^{t|x|} Q_i\left(-|x|\right).$$

It can be easily seen that, in a little bit more generality, the series (5.64) above converges uniformly on compact sets of (t,x) with $0 \leq t \leq T$ and $|x| \leq R$, for $f(x) = p_m(x)e^{-tx}$ where $p_m(x)$ is any polynomial of degree m. In particular, if $p_m(x) = \sum_{i=0}^{m} c_i^m Q_i(x)$ the previous bound becomes,

$$\sum_{k=0}^{\infty} |\langle Q_k, f\rangle_{\mathfrak{w}}||Q_k(x)|\pi(k) \leq e^{t|x|} \sum_{i=0}^{m} |c_i^m| Q_i\left(-|x|\right).$$

Remark 5.43. *Under certain regularity and growth assumptions on w at I^- and ∞, one can prove that the series in display* (5.64) *converges uniformly on compact intervals of $\mathfrak{I}$ for bounded variation functions f, such that their derivative satisfies a certain integrability condition (see in particular Theorem 4.17.2 of [109] and the references therein).*

We need one more property of functions of the form $f(x) = p_m(x)e^{-tx}$, namely that,

$$\langle Q_n, f\rangle_{\mathfrak{w}} \to 0 \text{ as } n \to \infty.$$

This can be seen as follows, by writing $p_m(x) = \sum_{i=0}^{m} \tilde{c}_i^m Q_i(x)\pi_i$ we have by (5.62),

$$\langle Q_n, f\rangle_{\mathfrak{w}} = \sum_{i=0}^{m} \tilde{c}_i^m p_t(n,i) \to 0 \text{ as } n \to \infty,$$

since, for any $i \in \mathbb{N}$ and $t \geq 0$, $p_t(n,i) \to 0$ as $n \to \infty$. Finally, we have the following relations between $\{Q_i\}_{i\geq 0}$ and their duals $\{\hat{Q}_i\}_{i\geq 0}$ (see [156] or section 6 of [92]),

$$\pi_{n+1}Q_{n+1}(x) = \hat{Q}_{n+1}(x) - \hat{Q}_n(x), \tag{5.65}$$

$$-x\hat{Q}_n(x) = \lambda_n\pi_n(Q_{n+1}(x) - Q_n(x)). \tag{5.66}$$

We are now in a position to prove the following result, which is modelled on and is essentially a generalization of Proposition 3.1 of [47]. It is what makes all subsequent calculations work.

Proposition 5.44. *1.* $\sum_{i=0}^{n}\pi_i Q_i(x) = \hat{Q}_n(x)$.

2. $\sum_{k=0}^{n-1}\hat{\pi}_k\hat{Q}_k(x) = \frac{\lambda_0}{x}(1 - Q_n(x))$.

3. $\langle\hat{Q}_n, f(0) - f\rangle_{\mathfrak{w}} = \sum_{k=n+1}^{\infty}\langle\pi_k Q_k, f\rangle_{\mathfrak{w}}$ *, for f in $L^2(\mathfrak{I},\mathfrak{w})$ so that series (5.64) converges pointwise at 0.*

4. $\sum_{k=n}^{\infty}\langle\hat{\pi}_k\hat{Q}_k, f\rangle_{\hat{\mathfrak{w}}} = \langle\hat{Q}_n, f\rangle_{\mathfrak{w}}$ *, for f in $L^2(\mathfrak{I},\mathfrak{w})$ so that $\langle Q_n, f\rangle_{\mathfrak{w}} \to 0$.*

Proof. To prove (1), note that by telescoping $\sum_{i=1}^{n}\pi_i Q_i(x) = \hat{Q}_n(x) - \hat{Q}_0(x) = \hat{Q}_n(x) - 1$ and that $\pi_0 Q_0(x) = 1$. To prove (2), first note,

$$\hat{\pi}(n)\hat{Q}_n(x) = \lambda(0)\left(\frac{Q_{n+1}(x) - Q_n(x)}{-x}\right)$$

and hence by summing,

$$\sum_{k=0}^{n-1}\hat{\pi}(k)\hat{Q}_k(x) = \lambda(0)\left(\frac{Q_n(x) - 1}{-x}\right).$$

To prove (3), observe that $\langle\hat{Q}_n, 1\rangle_{\mathfrak{w}} = \langle\sum_{i=0}^{n}\pi_i Q_i, 1\rangle_{\mathfrak{w}} = 1$. Also note that $Q_{n+1}(0) = Q_n(0) = \cdots = Q_0(0) = 1$ and thus from (1) we also get $\hat{Q}_n(0) = \sum_{k=0}^{n}\pi_k$. Moreover, by convergence of the orthogonal decomposition at 0 we have,

$$\langle\hat{Q}_n, f(0)\rangle_{\mathfrak{w}} = f(0) = \sum_{k=0}^{\infty}\langle Q_k, f\rangle_{\mathfrak{w}}Q_k(0)\pi(k) = \sum_{k=0}^{\infty}\langle\pi_k Q_k, f\rangle_{\mathfrak{w}},$$

$$\langle\hat{Q}_n, f\rangle_{\mathfrak{w}} = \sum_{k=0}^{n}\langle\pi_k Q_k, f\rangle_{\mathfrak{w}}.$$

Subtracting the two we get (3). In order to prove (4), we have,

$$\sum_{k=0}^{n-1}\langle\hat{\pi}_k\hat{Q}_k, f\rangle_{\hat{\mathfrak{w}}} = \langle\lambda(0)\left(\frac{Q_n(x) - 1}{-x}\right), f\rangle_{\hat{\mathfrak{w}}} = \langle 1 - Q_n, f\rangle_{\mathfrak{w}} \overset{n\to\infty}{\longrightarrow} \langle 1, f\rangle_{\mathfrak{w}},$$

where the limit holds by our assumption that $\langle Q_n, f\rangle_{\mathfrak{w}} \to 0$. Hence,

$$\sum_{k=n}^{\infty} \langle \hat{\pi}_k \hat{Q}_k, f\rangle_{\hat{\mathfrak{w}}} = \langle Q_n, f\rangle_{\mathfrak{w}}.$$

$\square$

5.7 Branching rules for multivariate Karlin-McGregor polynomials

For $\nu \in W^n$, we define the n-variate Karlin-McGregor polynomials by, with $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$,

$$\mathfrak{Q}_\nu(x) = \frac{\det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^n}{\det\left(x_j^{i-1}\right)_{i,j=1}^n} = \frac{\det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^n}{\Delta_n(x)}, \tag{5.67}$$

$$\hat{\mathfrak{Q}}_\nu(x) = \frac{\det\left(\hat{Q}_{\nu_i}(x_j)\right)_{i,j=1}^n}{\det\left(x_j^{i-1}\right)_{i,j=1}^n} = \frac{\det\left(\hat{Q}_{\nu_i}(x_j)\right)_{i,j=1}^n}{\Delta_n(x)}. \tag{5.68}$$

The polynomial systems, $\det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^n$ and $\det\left(\hat{Q}_{\nu_i}(x_j)\right)_{i,j=1}^n$ were first introduced by Karlin and McGregor, in their seminal study of intersection probabilities of birth and death chains in [94]. Some further properties were also presented in their subsequent brief note [95]. Observe that in particular, these multivariate polynomials are orthogonal in the continuous chamber $0 \le x_1 \le x_2 \cdots \le x_n$ (denoted $x \in W^n([0,\infty))$, with respect to the weights $\prod_{i=1}^n d\mathfrak{w}(x_i)\Delta_n^2(x)$ and $\prod_{i=1}^n d\hat{\mathfrak{w}}(x_i)\Delta_n^2(x)$ respectively.

Most importantly, we have the following *two-step* branching rules. The calculations below are in fact more or less implicitly done on page 1116 of [95].

Proposition 5.45.

$$\left.\frac{\det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^{n+1}}{\det\left(x_j^{i-1}\right)_{i,j=1}^{n+1}}\right|_{x_1=0} = \frac{(-1)^n}{\lambda_0^n} \sum_{k \in W^{n,n+1}(\nu)} \prod_{i=1}^n \hat{\pi}_{k_i} \frac{\det\left(\hat{Q}_{k_i}(x_{j+1})\right)_{i,j=1}^n}{\det\left(x_{j+1}^{i-1}\right)_{i,j=1}^n}, \tag{5.69}$$

$$\frac{\det\left(\hat{Q}_{\nu_i}(x_j)\right)_{i,j=1}^n}{\det\left(x_j^{i-1}\right)_{i,j=1}^n} = \sum_{k \in W^{n,n}(\nu)} \prod_{i=1}^n \pi_{k_i} \frac{\det\left(Q_{k_i}(x_j)\right)_{i,j=1}^n}{\det\left(x_j^{i-1}\right)_{i,j=1}^n}. \tag{5.70}$$

Proof. We prove (5.69) first. In the first equality below we make use of the fact that $Q_k(0) = 1$

and in the last one we make use of the relation $-x\hat{Q}(x) = \lambda_n \pi_n (Q_{n+1}(x) - Q_n(x))$.

$$
\begin{aligned}
\left.\frac{\det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^{n+1}}{\det\left(x_j^{i-1}\right)_{i,j=1}^{n+1}}\right|_{x_1=0} &= \frac{\det\left(Q_{\nu_{i+1}}(x_{j+1}) - Q_{\nu_i}(x_{j+1})\right)_{i,j=1}^{n}}{\det\left(x_{j+1}^{i-1}\right)_{i,j=1}^{n} \prod_{j=1}^{n} x_{j+1}} \\
&= \frac{\det\left(\frac{Q_{\nu_{i+1}}(x_{j+1}) - Q_{\nu_i}(x_{j+1})}{x_{j+1}}\right)_{i,j=1}^{n}}{\det\left(x_{j+1}^{i-1}\right)_{i,j=1}^{n}} \\
&= \sum_{k \in W^{n,n+1}(\nu)} \frac{\det\left(\frac{Q_{k_i+1}(x_{j+1}) - Q_{k_i}(x_{j+1})}{x_{j+1}}\right)_{i,j=1}^{n}}{\det\left(x_{j+1}^{i-1}\right)_{i,j=1}^{n}} \\
&= \sum_{k \in W^{n,n+1}(\nu)} \frac{\det\left(-\frac{\hat{\pi}_{k_i}}{\lambda_0} \hat{Q}_{k_i}(x_{j+1})\right)_{i,j=1}^{n}}{\det\left(x_{j+1}^{i-1}\right)_{i,j=1}^{n}}.
\end{aligned}
$$

In order to prove (5.70) we make use of part 1 of Proposition 5.44 so that (where we set $\nu_0 + 1 = 0$),

$$
\frac{\det\left(\hat{Q}_{\nu_i}(x_j)\right)_{i,j=1}^{n}}{\det\left(x_j^{i-1}\right)_{i,j=1}^{n}} = \frac{\det\left(\sum_{k_i=0}^{\nu_i} \pi_{k_i} Q_{k_i}(x_j)\right)_{i,j=1}^{n}}{\det\left(x_j^{i-1}\right)_{i,j=1}^{n}} = \frac{\det\left(\sum_{k_i=\nu_{i-1}+1}^{\nu_i} \pi_{k_i} Q_{k_i}(x_j)\right)_{i,j=1}^{n}}{\det\left(x_j^{i-1}\right)_{i,j=1}^{n}}.
$$

Note that, we can finally pull out the sum $\sum_{k \in W^{n,n}(\nu)}$ by multilinearity. □

Consider the functions,

$$
h_{n,n+1}(\nu, x) = (-1)^{\binom{n}{2}} \lambda_0^{\binom{n}{2}} \mathfrak{Q}_\nu(x), \text{ for } \nu \in W^{n+1}(\mathbb{N}) \text{ and } x \in W^{n+1}([0,\infty)), \tag{5.71}
$$

$$
h_{n,n}(\nu, x) = (-1)^{\binom{n-1}{2}} \lambda_0^{\binom{n-1}{2}} \hat{\mathfrak{Q}}_\nu(x), \text{ for } \nu \in W^{n}(\mathbb{N}) \text{ and } x \in W^{n}([0,\infty)) \tag{5.72}
$$

and define, for ν in W^{n+1} and W^n respectively,

$$
h_{n,n+1}(\nu) = h_{n,n+1}(\nu, 0), \tag{5.73}
$$

$$
h_{n,n}(\nu) = h_{n,n}(\nu, 0). \tag{5.74}
$$

Now, from the branching rules and our original intertwining relations from section 5.2.3 we prove the following:

Proposition 5.46. *$h_{n,n+1}$ and $h_{n,n}$ are positive harmonic functions for $n+1$ independent copies of $\mathcal{D}$-chains and n independent copies of $\hat{\mathcal{D}}$-chains in W^{n+1} and W^n respectively.*

Proof. Observe that, from the branching relations we get,

$$h_{n,n}(\nu) = (\Lambda_{n,n} h_{n-1,n})(\nu), \text{ for } \nu \in W^n(\mathbb{N}),$$
$$h_{n,n+1}(\nu) = (\Lambda_{n,n+1} h_{n,n})(\nu), \text{ for } \nu \in W^{n+1}(\mathbb{N})$$

and hence,

$$h_{n,n}(\nu) = (\Lambda_{n,n}\Lambda_{n-1,n}\cdots\Lambda_{1,1}\mathbf{1})(\nu), \text{ for } \nu \in W^n(\mathbb{N}),$$
$$h_{n,n+1}(\nu) = (\Lambda_{n,n+1}\Lambda_{n,n}\cdots\Lambda_{1,1}\mathbf{1})(\nu), \text{ for } \nu \in W^{n+1}(\mathbb{N}).$$

From relations (5.26) and (5.27) and the discussion around them, the conclusion is now evident. □

Remark 5.47. *In fact, some more general eigenfunction relations exist. For $x_1 < x_2 < \cdots < x_n \leq 0$ we have,*

$$(-1)^{\frac{n(n-1)}{2}} \det\left(Q_{\nu_i}(x_j)\right)_{i,j=1}^{n} > 0$$

and it can be readily checked that this is an eigenfunction of n independent $\mathcal{D}$-chains in W^n (see for example displays (19) and (30) respectively in [94]). These eigenfunctions can also be used to construct consistent dynamics and we will pursue this elsewhere.

Before continuing, we briefly recall some well known determinantal conditions for interlacing, namely representations of $\mathbf{1}\left(k \in W^{n,n+1}(\nu)\right)$ and $\mathbf{1}(k \in W^{n,n}(\nu))$ in terms of determinants. First of all, we have the following identity for $\mathbf{1}(y \in W^{n,n}(x))$,

$$\mathbf{1}(y_1 \leq x_1 < y_2 \leq \cdots \leq x_n) = \det\left(\mathbf{1}(y_i \leq x_j)\right)_{i,j=1}^{n}.$$

From this, by swapping x's and y's and putting $y_{n+1} = \infty$, or by declaring $y_{n+1} = \text{virt}$, a virtual variable and agreeing that $\mathbf{1}(x \leq \text{virt}) = 1$, we obtain the analogous identity for $\mathbf{1}\left(y \in W^{n,n+1}(x)\right)$,

$$\mathbf{1}(x_1 \leq y_1 < x_2 \leq \cdots \leq y_n < x_{n+1}) = \det\left(\mathbf{1}(x_i \leq y_j)\right)_{i,j=1}^{n+1}.$$

This can also be written as, after subtracting the last column from each of the rest,

$$\mathbf{1}(x_1 \leq y_1 < x_2 \leq \cdots < x_{n+1}) = \det\left(f_{i,j}\right)_{i,j=1}^{n+1},$$

where,

$$f_{i,j} = \begin{cases} -\mathbf{1}(x_i > y_j) & \text{if } j \leq n \\ 1 & \text{if } j = n+1 \end{cases}.$$

Thus, if we define,

$$
\begin{aligned}
\phi(i,j) &= \pi_i \mathbf{1}(i \le j),\\
\hat{\phi}(i,j) &= -\hat{\pi}_i \mathbf{1}(i < j),\\
\hat{\phi}(\mathrm{virt}, j) &= 1,
\end{aligned}
$$

then from Proposition 5.45, it is easy to see that:

Corollary 5.48. *The kernels $\Lambda^{h_{n,n+1}}_{n,n+1}(\nu,\cdot)$ and $\Lambda^{h_{n,n}}_{n,n}(\nu,\cdot)$, for any $\nu \in W^{n+1}$ and $\nu \in W^n$ respectively, that are defined by,*

$$
\Lambda^{h_{n,n}}_{n,n+1}(\nu,k) = \mathbf{1}\left(k \in W^{n,n+1}(\nu)\right) \frac{\prod_{i=1}^n \hat{\pi}_{k_i} h_{n,n}(k)}{h_{n,n+1}(\nu)} = \frac{\det\left(\hat{\phi}(k_i,\nu_j)\right)_{i,j=1}^{n+1} h_{n,n}(k)}{h_{n,n+1}(\nu)}, \tag{5.75}
$$

$$
\Lambda^{h_{n-1,n}}_{n,n}(\nu,k) = \mathbf{1}\left(k \in W^{n,n}(\nu)\right) \frac{\prod_{i=1}^n \pi_{k_i} h_{n-1,n}(k)}{h_{n,n}(\nu)} = \frac{\det\left(\phi(k_i,\nu_j)\right)_{i,j=1}^{n} h_{n-1,n}(k)}{h_{n,n}(\nu)}, \tag{5.76}
$$

are Markov.

Finally, denoting by $\left(P^{h_{n,n+1}}_{n+1}(t); t \ge 0\right)$ and $\left(\hat{P}^{h_{n,n}}_{n}(t); t \ge 0\right)$ the Karlin-McGregor semigroups associated with $n+1$ $\mathcal{D}$-chains and n $\hat{\mathcal{D}}$-chains, h-transformed by $h_{n,n+1}$ and $h_{n,n}$ respectively, we immediately get the following corollary of Theorems 5.18 and 5.20.

Corollary 5.49. *For $t \ge 0$, we have the intertwining relations,*

$$
P^{h_{n,n+1}}_{n+1}(t)\Lambda^{h_{n,n}}_{n,n+1} = \Lambda^{h_{n,n}}_{n,n+1}\hat{P}^{h_{n,n}}_{n}(t), \tag{5.77}
$$

$$
\hat{P}^{h_{n,n}}_{n}(t)\Lambda^{h_{n-1,n}}_{n,n} = \Lambda^{h_{n-1,n}}_{n,n}P^{h_{n-1,n}}_{n}(t). \tag{5.78}
$$

5.8 Coherent measures

We now move on towards defining, in displays (5.81) and (5.82), measures denoted by $\mathcal{M}^{\psi}_{n,n+1}$ and $\mathcal{M}^{\psi}_{n,n}$, depending on a function ψ, that are coherent with respect to the Markov links $\Lambda^{h_{n,n}}_{n,n+1}$ and $\Lambda^{h_{n-1,n}}_{n,n}$. We first need some definitions and technical preliminaries.

Consider the Taylor remainder for a function f, that is $(n-1)$-times differentiable at 0, given by,

$$
\mathsf{R}^f_n(x) = \begin{cases} f(x) & n \le 0 \\ f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k & n \ge 1 \end{cases}.
$$

Now, define for f that is $(j-n)$-times or $(j-(n+1))$-times continuously differentiable at 0 respectively, the following functions on $\mathbb{N}$, $\Psi^{n,n+1}_{n+1-j}(\cdot)$ and $\Psi^{n,n}_{n-j}(\cdot)$ (their dependence on

f will be suppressed),

$$\Psi^{n,n+1}_{n+1-j}(i) = \langle \pi_i Q_i, (-x)^{n+1-j} \mathsf{R}^f_{j-(n+1)} \rangle_{\mathfrak{w}},\ i \in \mathbb{N}, \tag{5.79}$$

$$\Psi^{n,n}_{n-j}(i) = \langle \hat{\pi}_i \hat{Q}_i, (-x)^{n-j} \mathsf{R}^f_{j-n} \rangle_{\hat{\mathfrak{w}}},\ i \in \mathbb{N}. \tag{5.80}$$

We also define the discrete convolution for functions $h_1, h_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ and $h_3 : \mathbb{N} \to \mathbb{C}$ as follows,

$$\left(h_1 * h_2\right)(u,v) = \sum_{k\geq 0} h_1(u,k) h_2(k,v),$$

$$\left(h_1 * h_3\right)(u) = \sum_{k\geq 0} h_1(u,k) h_3(k).$$

The lemma below states that, alternating convolutions of ϕ and $\hat{\phi}$ with $\Psi^{n,n}_{n-j}$ and $\Psi^{n,n+1}_{n+1-j}$ respectively are nicely consistent. This will be useful in the computations performed in Proposition 5.53 that proves that the measures introduced below are indeed coherent.

Lemma 5.50. *Assume that $f(x) = p(x)e^{-tx}$, where $p(x)$ is a fixed polynomial of arbitrary degree. Then, we have,*

1. $\left(\phi * \Psi^{n,n}_{n-j}\right)(i) = \Psi^{n-1,n}_{n-j}(i)$.
2. $\left(\hat{\phi} * \Psi^{n,n+1}_{n+1-j}\right)(i) = -\lambda_0 \Psi^{n,n}_{n-j}(i)$.

Proof. To prove (1) note,

$$\begin{aligned}
\left(\phi * \Psi^{n,n}_{n-j}\right)(i) &= \sum_{k\geq 0} \pi_i \mathbf{1}(i \leq k) \Psi^{n,n}_{n-j}(k) \\
&= \sum_{k\geq i} \pi_i \langle \hat{\pi}_k \hat{Q}_k, (-x)^{n-j} \mathsf{R}^f_{j-n} \rangle_{\hat{\mathfrak{w}}} \\
&= \pi_i \langle Q_i, (-x)^{n-j} \mathsf{R}^f_{j-n} \rangle_{\mathfrak{w}} = \Psi^{n-1,n}_{n-j}(i).
\end{aligned}$$

Now to prove (2) first observe that with $\mathsf{T}^f_m(x) = (-x)^{-m} \mathsf{R}^f_m(x)$ then, $\mathsf{T}^f_m(0) = \lim_{x\to 0} \mathsf{T}^f_m(x) = \frac{f^{(m)}(0)}{m!}(-1)^m$ and so,

$$\begin{aligned}
\left(\hat{\phi} * \Psi^{n,n+1}_{n+1-j}\right)(i) &= -\sum_{k\geq 0} \hat{\pi}_i \mathbf{1}(i < k) \Psi^{n,n+1}_{n+1-j}(k) \\
&= \sum_{k\geq i+1} \hat{\pi}_i \langle \pi_k Q_k, (-x)^{n+1-j} \mathsf{R}^f_{j-(n+1)} \rangle_{\mathfrak{w}} \\
&= -\hat{\pi}_i \langle \hat{Q}_i, ((-x)^{n+1-j} \mathsf{R}^f_{j-(n+1)})(0) - (-x)^{n+1-j} \mathsf{R}^f_{j-(n+1)} \rangle_{\mathfrak{w}}.
\end{aligned}$$

Moreover, since $d\hat{\mathfrak{w}} = \frac{x d\mathfrak{w}}{\lambda(0)}$ and $\frac{1}{x}\left(((-x)^{n+1-j} \mathsf{R}^f_{j-(n+1)})(0) - (-x)^{n+1-j} \mathsf{R}^f_{j-(n+1)}\right) = (-x)^{n-j} \mathsf{R}^f_{n-j}$ we get,

$$\left(\hat{\phi} * \Psi^{n,n+1}_{n+1-j}\right)(i) = -\lambda_0 \langle \hat{\pi}_i \hat{Q}_i, (-x)^{n-j} \mathsf{R}^f_{j-n} \rangle_{\hat{\mathfrak{w}}}.$$

□

Remark 5.51. *Of course, the condition that $f(x) = p(x)e^{-tx}$ is unnecessarily restrictive. All that is needed, other than the necessary differentiability assumptions on f, in order to prove (1) is that $\langle Q_k, (-x)^{n-j} \mathsf{R}^f_{j-n} \rangle_{\mathfrak{w}} \to 0$ as $k \to \infty$ and for (2) that the orthogonal decomposition of T^f_m converges pointwise at 0.*

We now, define the *coherent measures* $\mathcal{M}^{\psi}_{n,n+1}$ and $\mathcal{M}^{\psi}_{n,n}$, for ψ in $L^2(\mathfrak{I}, \mathfrak{w})$ or $L^2(\mathfrak{I}, \hat{\mathfrak{w}})$ respectively as follows,

$$\mathcal{M}^{\psi}_{n,n+1}(\nu) = \frac{(-1)^{\binom{n}{2}}}{\lambda_0^{\binom{n}{2}}} \det\left(\langle \pi_{\nu_i} Q_{\nu_i}, (-x)^{n+1-j} \psi \rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1} h_{n,n+1}(\nu), \text{ for } \nu \in W^{n+1}, \tag{5.81}$$

$$\mathcal{M}^{\psi}_{n,n}(\nu) = \frac{(-1)^{\binom{n-1}{2}}}{\lambda_0^{\binom{n-1}{2}}} \det\left(\langle \hat{\pi}_{\nu_i} \hat{Q}_{\nu_i}, (-x)^{n-j} \psi \rangle_{\hat{\mathfrak{w}}}\right)^{n}_{i,j=1} h_{n,n}(\nu), \text{ for } \nu \in W^{n}. \tag{5.82}$$

Note that, by simply unpacking the notation and observing that the powers of (-1)'s actually cancel out, these can be written as,

$$\mathcal{M}^{\psi}_{n,n+1}(\nu_1, \cdots, \nu_{n+1}) = \frac{1}{\lambda_0^{\binom{n}{2}}} \det\left(\int_{\mathfrak{I}} \pi_{\nu_i} Q_{\nu_i}(x) x^{n+1-j} \psi(x) d\mathfrak{w}(x)\right)^{n+1}_{i,j=1} h_{n,n+1}(\nu_1, \cdots, \nu_{n+1}),$$

$$\mathcal{M}^{\psi}_{n,n}(\nu_1, \cdots, \nu_n) = \frac{1}{\lambda_0^{\binom{n-1}{2}}} \det\left(\int_{\mathfrak{I}} \hat{\pi}_{\nu_i} \hat{Q}_{\nu_i}(x) x^{n-j} \psi(x) d\hat{\mathfrak{w}}(x)\right)^{n}_{i,j=1} h_{n,n}(\nu_1, \cdots, \nu_n).$$

The measures $\mathcal{M}^{\psi}$ are real (not necessarily positive) measures and as we see in Lemma 5.52 below their mass is explicit. Moreover, Lemma 5.52, shows that the "generating functions" (with respect to the corresponding multivariate orthogonal polynomials) of these measures are *multiplicative*. This property, under some extra assumptions (see Appendix), implies that these coherent measures, when they are positive and normalized to be probability measures (see subsection 5.9.2), are in fact *extremal* (and thus, they correspond to points of the boundary of the branching graph coming from the alternating construction, see subsection 5.4.3).

Lemma 5.52. *With $\star = n, n+1$, let $\psi \in L^2$ be such that each of the functions $\{(-x)^{n+1-i}\psi(x)\}_{i=1}^{n+1}$ has an orthogonal decomposition converging pointwise at the points $\{x_j\}_{j=1}^{n+1}$. Then,*

$$\sum_{\nu \in W^{\star}} \mathcal{M}^{\psi}_{n,\star}(\nu) \frac{h_{n,\star}(x,\nu)}{h_{n,\star}(\nu)} = \prod_{i=1}^{\star} \psi(x_i), \tag{5.83}$$

where the functions $h_{n,\star}$ where defined in (5.71), (5.72). In particular, the measures $\mathcal{M}^{\psi}_{n,\star}$ have mass $\psi(0)^{\star}$. Moreover, if $\psi \equiv 1$ then $\mathcal{M}^{\psi}_{n,\star}(\nu) = \mathbf{1}(\nu = (0, \cdots, \star - 1))$.

Proof. We apply the Cauchy-Binet formula (for infinite sums, see for example Lemma 2.1 of [47]), to obtain with $\star = n+1$ (the case $\star = n$ is exactly the same with only changes in notation),

$$\sum_{\nu \in W^{n+1}} \mathcal{M}^{\psi}_{n,n+1}(\nu) \frac{h_{n,n+1}(x,\nu)}{h_{n,n+1}(\nu)} = \frac{\det\left(\sum_{k\geq 0} \langle \pi_k Q_k, (-x)^{n+1-i} \mathsf{R}^{\psi}_{i-(n+1)} \rangle_{\mathfrak{w}} Q_k(x_j)\right)_{i,j=1}^{n+1}}{\det\left(x_i^{j-1}\right)_{i,j=1}^{n+1}}$$

$$= \frac{\det\left((-x_j)^{n+1-i}\psi(x_j)\right)_{i,j=1}^{n+1}}{\det\left(x_i^{j-1}\right)_{i,j=1}^{n+1}} = \prod_{i=1}^{n+1} \psi(x_i).$$

We have also used the fact that,

$$\frac{\det\left((-x_i)^{n+1-j}\right)_{i,j=1}^{n+1}}{\det\left(x_i^{j-1}\right)_{i,j=1}^{n+1}} = (-1)^{\binom{n}{2}} \frac{\det\left(x_i^{n+1-j}\right)_{i,j=1}^{n+1}}{\det\left(x_i^{j-1}\right)_{i,j=1}^{n+1}} = (-1)^{\binom{n}{2}}(-1)^{\lfloor \frac{n+1}{2} \rfloor} \equiv 1.$$

Moreover, we have,

$$\mathcal{M}^{\psi}_{n,n+1}(0, \cdots, n) = \left.\frac{\det\left(\sum_{k\geq 0} \langle \pi_k Q_k, (-x)^{n+1-i} \rangle_{\mathfrak{w}} Q_k(x_j)\right)_{i,j=1}^{n+1}}{\det\left(x_i^{j-1}\right)_{i,j=1}^{n+1}}\right|_{x_1,\cdots,x_n=0}$$

$$= \left.\frac{\det\left((-x_j)^{n+1-i}\right)_{i,j=1}^{n+1}}{\det\left(x_i^{j-1}\right)_{i,j=1}^{n+1}}\right|_{x_1,\cdots,x_n=0} = 1.$$

□

Our interest in these measures, as already anticipated, stems from the fact that they are coherent/consistent with respect to the intertwining kernels.

Proposition 5.53. *Let $\psi(x) = p(x)e^{-tx}$, where $p(x)$ is a polynomial of arbitrary degree. Then with $k \in W^n$,*

$$\mathcal{M}^{\psi}_{n,n}(k) = (\mathcal{M}^{\psi}_{n,n+1}\Lambda^{h_{n,n}}_{n,n+1})(k)\text{ , for } \psi(0) = 1, \tag{5.84}$$

$$\mathcal{M}^{\psi}_{n-1,n}(k) = (\mathcal{M}^{\psi}_{n,n}\Lambda^{h_{n-1,n}}_{n,n})(k). \tag{5.85}$$

Proof. We prove (5.85) first, using the Cauchy-Binet formula for the passage to the second equality,

$$\begin{aligned}\sum_{\nu\in W^n}\mathcal{M}^{\psi}_{n,n}(\nu)\Lambda^{h_{n-1,n}}_{n,n}(\nu,k) &= \frac{(-1)^{\binom{n-1}{2}}}{\lambda_0^{\binom{n-1}{2}}}h_{n-1,n}(k)\sum_{\nu\in W^n}\det\left(\phi(k_i,\nu_j)\right)_{i,j=1}^n\det\left(\langle\hat{\pi}_{\nu_i}\hat{Q}_{\nu_i},(-x)^{n-j}\mathsf{R}^{\psi}_{j-n}\rangle_{\hat{\mathfrak{w}}}\right)_{i,j=1}^n\\ &= \frac{(-1)^{\binom{n-1}{2}}}{\lambda_0^{\binom{n-1}{2}}}h_{n-1,n}(k)\det\left((\phi*\Psi^{n,n}_{n-j})(k_i)\right)_{i,j=1}^n\\ &= \frac{(-1)^{\binom{n-1}{2}}}{\lambda_0^{\binom{n-1}{2}}}h_{n-1,n}(k)\det\left(\Psi^{n-1,n}_{n-j}(k_i)\right)_{i,j=1}^n = \mathcal{M}^{\psi}_{n-1,n}(k).\end{aligned}$$

We now turn to the proof of (5.84) and calculate, again using the Cauchy-Binet formula for the second equality,

$$\begin{aligned}\sum_{\nu\in W^{n+1}}\mathcal{M}^{\psi}_{n,n+1}(\nu)\Lambda^{h_{n,n}}_{n,n}(\nu,k) &= \frac{(-1)^{\binom{n}{2}}}{\lambda_0^{\binom{n}{2}}}h_{n,n}(k)\sum_{\nu\in W^n}\det\left(\hat{\phi}(k_i,\nu_j)\right)_{i,j=1}^{n+1}\det\left(\langle\pi_{\nu_i}Q_{\nu_i},(-x)^{n-j}\mathsf{R}^{\psi}_{j-n}\rangle_{\mathfrak{w}}\right)_{i,j=1}^{n+1}\\ &= \frac{(-1)^{\binom{n}{2}}}{\lambda_0^{\binom{n}{2}}}h_{n,n}(k)\det\left((\hat{\phi}*\Psi^{n,n+1}_{n+1-j})(k_i)\right)_{i,j=1}^{n+1}\\ &= \frac{(-1)^{\binom{n-1}{2}}}{\lambda_0^{\binom{n-1}{2}}}h_{n,n}(k)\det\left(\Psi^{n,n}_{n-j}(k_i)\right)_{i,j=1}^n = \mathcal{M}^{\psi}_{n,n}(k).\end{aligned}$$

The penultimate equality, follows from $\left(\hat{\phi}*\Psi^{n,n+1}_{n+1-j}\right)(i) = -\lambda_0\Psi^{n,n}_{n-j}(i)$ and the fact that the last row of $\{(\hat{\phi}*\Psi^{n,n+1}_{n+1-j})(k_i)\}_{i,j=1}^{n+1}$ is given by, with $k_{n+1}=\mathrm{virt}$ (recall for $j\le n+1$ that $\mathsf{R}^{\psi}_{j-(n+1)}=\psi$),

$$(\hat{\phi}*\Psi^{n,n+1}_{n+1-j})(\mathrm{virt}) = \sum_{i\ge 0}\Psi^{n,n+1}_{n+1-j}(i) = \sum_{i\ge 0}\langle\pi_iQ_i,(-x)^{n+1-j}\mathsf{R}^{\psi}_{j-(n+1)}\rangle_{\mathfrak{w}}Q_i(0) = ((-x)^{n+1-j}\psi)(0) = \delta_{j,n+1},$$

where we have assumed $\psi(0)=1$ and also used the fact that $Q_i(0)=1$. □

Remark 5.54. *Again, conditions on ψ can be relaxed c.f. Remark 5.51.*

5.9 Evolution of coherent measures

5.9.1 Evolution operators for coherent measures and their basic properties

We now define some kind of evolution operators acting on the coherent measures, that generalize the h-transformed Karlin-McGregor semigroups. For ψ in $L^2(\mathfrak{I},\mathfrak{w})$ and $L^2(\mathfrak{I},\hat{\mathfrak{w}})$

respectively, define $\mathfrak{P}^{\psi}_{n,n+1}$ and $\mathfrak{P}^{\psi}_{n,n}$ by,

$$\mathfrak{P}^{\psi}_{n,n+1}(k,\nu)=\frac{h_{n,n+1}(\nu)}{h_{n,n+1}(k)}\det\left(\langle Q_{k_i},\pi_{\nu_j}Q_{\nu_j}\psi\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1},\ \text{for } k,\nu\in W^{n+1}, \tag{5.86}$$

$$\mathfrak{P}^{\psi}_{n,n}(k,\nu)=\frac{h_{n,n}(\nu)}{h_{n,n}(k)}\det\left(\langle \hat{Q}_{k_i},\hat{\pi}_{\nu_j}\hat{Q}_{\nu_j}\psi\rangle_{\hat{\mathfrak{w}}}\right)^{n}_{i,j=1},\ \text{for } k,\nu\in W^{n}. \tag{5.87}$$

Note that,

$$\mathfrak{P}^{\psi}_{\bullet,\star}(k_0,\nu)=\mathcal{M}^{\psi}_{\bullet,\star}(\nu),\ \text{where } k_0=(0,1,\cdots,\star-1). \tag{5.88}$$

This is because, by row and column operations both sides are the same up to a multiplicative constant and since, from the following lemma they both sum to $\psi(0)^{\star}$, they must in fact be equal.

Moreover, observe that by (5.62) for $\psi(x)=\phi_t(x)=e^{-tx}$ then $\left(\mathfrak{P}^{\phi_t}_{n,n+1};t\geq 0\right)$ and $\left(\mathfrak{P}^{\phi_t}_{n,n};t\geq 0\right)$ are exactly the h-transformed Karlin-McGregor semigroups $\left(P^{h_{n,n+1}}_{n+1}(t);t\geq 0\right)$ and $\left(\hat{P}^{h_{n,n}}_{n}(t);t\geq 0\right)$ respectively. We will now study their properties. The non-trivial issue of positivity will be dealt with at the end of this subsection. First, we have the following lemma regarding their normalization.

Lemma 5.55. *If, ψ is such that its orthogonal decomposition converges pointwise in a neighbourhood of 0, we then have,*

$$\sum_{\nu\in W^{n+1}}\mathfrak{P}^{\psi}_{n,n+1}(k,\nu)=\psi(0)^{n+1},\ \forall k\in W^{n+1},$$
$$\sum_{\nu\in W^{n}}\mathfrak{P}^{\psi}_{n,n}(k,\nu)=\psi(0)^{n},\ \forall k\in W^{n}.$$

Proof. We only prove the first equality, as the second is analogous,

$$\begin{aligned}\sum_{\nu\in W^{n+1}}\mathfrak{P}^{\psi}_{n,n+1}(k,\nu)&=\frac{1}{h_{n,n+1}(k)}\sum_{\nu\in W^{n+1}}\det\left(\langle Q_{k_i},\pi_{\nu_i}Q_{\nu_i}\psi\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1}(-1)^{\binom{n}{2}}\lambda_0^{\binom{n}{2}}\left.\frac{\det\left(Q_{\nu_i}(x_j)\right)^{n+1}_{i,j=1}}{\det\left(x_j^{i-1}\right)^{n+1}_{i,j=1}}\right|_{x_1,\cdots,x_{n+1}=0}\\&=\frac{(-1)^{\binom{n}{2}}\lambda_0^{\binom{n}{2}}}{h_{n,n+1}(k)}\left.\frac{\det\left(\sum_{m\geq 0}\langle Q_{k_i},\pi_m Q_m\psi\rangle_{\mathfrak{w}}Q_m(x_j)\right)^{n+1}_{i,j=1}}{\det\left(x_j^{i-1}\right)^{n+1}_{i,j=1}}\right|_{x_1,\cdots,x_{n+1}=0}\\&=\frac{(-1)^{\binom{n}{2}}\lambda_0^{\binom{n}{2}}}{h_{n,n+1}(k)}\left.\frac{\det\left(Q_{k_i}(x_j)\right)^{n+1}_{i,j=1}}{\det\left(x_j^{i-1}\right)^{n+1}_{i,j=1}}\prod_{i=1}^{n+1}\psi(x_i)\right|_{x_1,\cdots,x_{n+1}=0}=\psi(0)^{n+1}.\end{aligned}$$

□

The simple, but important proposition below, describes the evolution of coherent measures. Its proof is an easy consequence of the Cauchy-Binet formula and of uniform convergence of the orthogonal decomposition on compact sets for functions of the form $p(x)e^{-tx}$, with $p(x)$ a polynomial.

Proposition 5.56. *Assume $\mathfrak{I}$ is compact or equivalently $I^+ < \infty$ and moreover suppose $\psi_1(x) = p_1(x)e^{-t_1x}$ and $\psi_2(x) = p_2(x)e^{-t_2x}$, where p_1, p_2 are arbitrary polynomials and $t_1, t_2 \geq 0$. We then have the following equalities,*

$$\sum_{k\in W^{n+1}} \mathcal{M}^{\psi_1}_{n,n+1}(k)\mathfrak{P}^{\psi_2}_{n,n+1}(k,\nu) = \mathcal{M}^{\psi_1\psi_2}_{n,n+1}(\nu),\ \forall \nu \in W^{n+1}, \tag{5.89}$$

$$\sum_{k\in W^{n}} \mathcal{M}^{\psi_1}_{n,n}(k)\mathfrak{P}^{\psi_2}_{n,n}(k,\nu) = \mathcal{M}^{\psi_1\psi_2}_{n,n}(\nu),\ \forall \nu \in W^{n}. \tag{5.90}$$

Proof. We only prove (5.89), as (5.90) is completely analogous. The passage to the second equality below first uses the Cauchy-Binet formula and secondly the uniform convergence of the orthogonal decomposition on compacts, in order to justify the interchange $\sum\langle\cdot,\cdot\rangle_{\mathfrak{w}} = \langle\sum\cdot,\cdot\rangle_{\mathfrak{w}}$, of summation and integration,

$$\begin{aligned}\sum_{k\in W^{n+1}} \mathcal{M}^{\psi_1}_{n,n+1}(k)\mathfrak{P}^{\psi_2}_{n,n+1}(k,\nu) &= (-1)^{\binom{n}{2}}\lambda_0^{\binom{n}{2}}h_{n,n+1}(\nu)\sum_{k\in W^{n+1}} \det\left(\langle \pi_{k_i}Q_{k_i}, (-x)^{n+1-j}\psi_1\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1} \det\left(\langle Q_{k_i}, \pi_{\nu_j}Q_{\nu_j}\psi_2\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1}\\ &= (-1)^{\binom{n}{2}}\lambda_0^{\binom{n}{2}}h_{n,n+1}(\nu)\det\left(\langle\sum_{m\geq 0}\langle \pi_m Q_m, \pi_{\nu_i}Q_{\nu_i}\psi_2\rangle_{\mathfrak{w}}Q_m, (-x)^{n+1-j}\psi_1\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1}\\ &= (-1)^{\binom{n}{2}}\lambda_0^{\binom{n}{2}}h_{n,n+1}(\nu)\det\left(\langle \pi_{\nu_i}Q_{\nu_i}, (-x)^{n+1-j}\psi_1\psi_2\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1} = \mathcal{M}^{\psi_1\psi_2}_{n,n+1}(\nu).\end{aligned}$$

□

Remark 5.57. *In fact, the argument above gives,*

$$\sum_{k\in W^{n+1}} \mathfrak{P}^{\psi_1}_{n,n+1}(\mu,k)\mathfrak{P}^{\psi_2}_{n,n+1}(k,\nu) = \mathfrak{P}^{\psi_1\psi_2}_{n,n+1}(\mu,\nu),\ \forall \mu,\nu \in W^{n+1},$$

$$\sum_{k\in W^{n}} \mathfrak{P}^{\psi_1}_{n,n}(\mu,k)\mathfrak{P}^{\psi_2}_{n,n}(k,\nu) = \mathfrak{P}^{\psi_1\psi_2}_{n,n}(\mu,\nu),\ \forall \mu,\nu \in W^{n}.$$

Then (5.89) and (5.90) become a consequence of (5.88).

Remark 5.58. *The assumptions that $\mathfrak{I}$ is compact and that ψ_1, ψ_2 are of the special form $p(x)e^{-tx}$ could of course be removed as long as the interchange of summation and integration in the second equality above can be justified.*

Finally, we give a linear algebraic proof of the following intertwining relations. Although, we have already obtained these equalities in the special case $\psi_t(x) = e^{-tx}$ in Corollary 5.49 by other means and for general functions ψ will not be used in the sequel; we decided to present it, since it sheds some light on the relations between the dual Karlin-McGregor polynomials that are essential for these commutation relations to hold.

Proposition 5.59. *Let ψ be as in the statement of Proposition 5.56 and moreover assume $\psi(0) = 1$. Then,*

$$\begin{aligned}\mathfrak{P}^{\psi}_{n,n+1}\Lambda^{h_{n,n}}_{n,n+1} &= \Lambda^{h_{n,n}}_{n,n+1}\mathfrak{P}^{\psi}_{n,n},\\ \mathfrak{P}^{\psi}_{n,n}\Lambda^{h_{n-1,n}}_{n,n} &= \Lambda^{h_{n-1,n}}_{n,n}\mathfrak{P}^{\psi}_{n-1,n}.\end{aligned}$$

Proof. We only prove the first relation, as the second is analogous. Observe that (noting also that the dummy variable on the left is $(n+1)$-dimensional while on the left n-dimensional),

$$\sum_{z\in W^{n+1}}\mathfrak{P}^{\psi}_{n,n+1}(k,z)\Lambda^{h_{n,n}}_{n,n+1}(z,\nu) = \sum_{z\in W^{n}}\Lambda^{h_{n,n}}_{n,n+1}(k,z)\mathfrak{P}^{\psi}_{n,n}(z,\nu),$$

is equivalent to,

$$\sum_{z\in W^{n+1}}\det\left(\langle Q_{k_i},\pi_{z_j}Q_{z_j}\psi\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1}\det\left(\hat{\phi}(\nu_j,z_i)\right)^{n+1}_{i,j=1} = \sum_{z\in W^{n}}\det\left(\hat{\phi}(z_j,k_i)\right)^{n+1}_{i,j=1}\det\left(\langle \hat{Q}_{z_i},\hat{\pi}_{\nu_j}\hat{Q}_{\nu_j}\psi\rangle_{\hat{\mathfrak{w}}}\right)^{n}_{i,j=1}.$$

The left hand side, by the Cauchy-Binet formula is equal to,

$$\det\left(\langle Q_{k_i},\sum_{z\geq 0}\pi_z Q_z\hat{\phi}(\nu_j,z)\psi\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1}.$$

For $j\leq n$, the entries of the matrix are given by (recall that $Q_{k_i}(0)=\psi(0)=1$),

$$\sum_{z=\nu_j+1}\langle \pi_z Q_z,-\hat{\pi}_{\nu_j}Q_{k_i}\psi\rangle_{\mathfrak{w}} = \langle \hat{Q}_{\nu_j},\hat{\pi}_{\nu_j}Q_{k_i}\psi-\hat{\pi}_{\nu_j}Q_{k_i}(0)\psi(0)\rangle_{\mathfrak{w}} = \langle \hat{Q}_{\nu_j},\hat{\pi}_{\nu_j}Q_{k_i}\psi\rangle_{\mathfrak{w}} - \langle \hat{Q}_{\nu_j},\hat{\pi}_{\nu_j}\rangle_{\mathfrak{w}} = \mathfrak{a}_{ij}+\mathfrak{b}_j.$$

While, the entries of the last column $j=n+1$ are,

$$\langle Q_{k_i},\sum_{z\geq 0}\pi_z Q_z\psi\rangle_{\mathfrak{w}} = Q_{k_i}(0)\psi(0) = 1.$$

To work on the right hand side, we first expand $\det\left(\hat{\phi}(z_j,k_i)\right)^{n+1}_{i,j=1}$ in the last column which consists of all 1's. The l^{th}-summand in this expansion is given by,

$$\begin{aligned}&(-1)^{n+1+l}\sum_{z\in W^n}\det\left(\hat{\phi}(z_j,k_i)\right)_{1\leq i\neq l\leq n+1,1\leq j\leq n}\det\left(\langle \hat{Q}_{z_i},\hat{\pi}_{\nu_j}\hat{Q}_{\nu_j}\psi\rangle_{\hat{\mathfrak{w}}}\right)^{n}_{i,j=1}\\ &\quad= (-1)^{n+1+l}\det\left(\sum_{z\geq 0}\hat{\phi}(z,k_i)\langle \hat{Q}_z,\hat{\pi}_{\nu_j}\hat{Q}_{\nu_j}\psi\rangle_{\hat{\mathfrak{w}}}\right)_{1\leq i\neq l\leq n+1,1\leq j\leq n}.\end{aligned}$$

The entries of the matrix in the determinant are given by,

$$-\langle\sum_{z=0}^{k_i-1}\hat{\pi}_z\hat{Q}_z,\hat{\pi}_{\nu_j}\hat{Q}_{\nu_j}\psi\rangle_{\hat{\mathfrak{w}}}=\langle(Q_{k_i}-1)\frac{\lambda_0}{x},\hat{\pi}_{\nu_j}\hat{Q}_{\nu_j}\psi\rangle_{\hat{\mathfrak{w}}}=\langle(Q_{k_i}-1),\hat{\pi}_{\nu_j}\hat{Q}_{\nu_j}\psi\rangle_{\mathfrak{w}}=\mathfrak{a}_{ij}+\mathfrak{c}_j.$$

Now, by summing over l, we obtain the determinant of an $(n+1)\times(n+1)$ matrix with the last column being all 1's and the other entries being $\mathfrak{a}_{ij}+\mathfrak{c}_j$.

By column operations, more precisely by subtracting a multiple of the last all 1's column from the each of the rest, the equality of the left and right hand sides is immediate.

□

Remark 5.60. *Proposition 5.53 can also be seen as a corollary of Proposition 5.59 using (5.88).*

5.9.2 Positivity of evolution operators and coherent measures

We now arrive at the question of positivity of the coherent measures. It will in fact be easier to consider a more general problem, namely to address this question first for the evolution operators.

As already observed by (5.62), for $\psi(z)=\phi_t(z)=e^{-tz}$ the determinants $\det\left(\langle Q_{k_i},\pi_{\nu_i}Q_{\nu_i}\phi_t\rangle_{\mathfrak{w}}\right)_{i,j=1}^{n+1}$ and $\det\left(\langle\hat{Q}_{k_i},\hat{\pi}_{\nu_i}\hat{Q}_{\nu_i}\phi_t\rangle_{\hat{\mathfrak{w}}}\right)_{i,j=1}^{n}$ are exactly the transition densities of the Karlin-McGregor semigroups associated to $n+1$ birth and death chains with generator $\mathcal{D}$ and n birth and death chains with generator $\hat{\mathcal{D}}$ respectively, killed when they collide and so they are positive. Hence, since $h_{n,n}$ and $h_{n,n+1}$ are positive as well we obtain:

Lemma 5.61. $\mathfrak{P}^{\phi_t}_{n,n+1}$ *and* $\mathfrak{P}^{\phi_t}_{n,n}$ *are positive,* $\forall t\geq 0$.

Our goal now, is to find conditions on a so that with $\psi_a(z)=1-az$ the operator $\mathfrak{P}^{\psi_a}_{n,n+1}$ is positive. We make use of an argument found in Proposition 5.1 of [47], that is recalled briefly here (see Proposition 5.1 part (4) of [47], in particular the paragraph between equations (23) and (24) therein, for the details). Our computations below, are quite simple (compared to [47], although we do follow the same argument) taking advantage of the relation between the normalization constants and the rates of the chain. First, we calculate for $i,j\in\mathbb{N}$,

$$\begin{aligned}\langle Q_i,\pi_jQ_j(1-az)\rangle_{\mathfrak{w}}&=\delta_{i,j}+a\pi_j\langle Q_i,\mu_jQ_{j-1}-(\lambda_j+\mu_j)Q_j+\lambda_jQ_{j+1}\rangle_{\mathfrak{w}}\\&=\delta_{i,j}+a\delta_{i,j-1}\frac{1}{\pi_{j-1}}\pi_j\mu_j-a(\lambda_j+\mu_j)\delta_{i,j}+a\delta_{i,j+1}\lambda_j\frac{1}{\pi_{j+1}}\pi_j\\&=\delta_{i,j}+a\lambda_{j-1}\delta_{i,j-1}-a(\mu_j+\lambda_j)\delta_{i,j}+a\mu_{j+1}\delta_{i,j+1},\end{aligned}$$

since $\frac{\pi_j}{\pi_{j-1}}=\frac{\lambda_{j-1}}{\mu_j}$.

We now, reduce the problem as in Proposition 5.1 of [47]. First, note that if $y_i>x_i+1$ for some i then we get $\det\left(\langle Q_{x_i},\pi_{y_j}Q_{y_j}\psi\rangle_{\mathfrak{w}}\right)_{i,j=1}^{n}=0$, since the resulting matrix has a 2×2 block form consisting of an off diagonal block of 0's and a diagonal block of 0's and the same happens for $x_i > y_i + 1$. Thus, we must have $|x_i - y_i| \leq 1$ and we can further restrict to the case $|x_i - x_{i+1}| \leq 1$, for otherwise $\det\left(\langle Q_{x_i}, \pi_{y_j} Q_{y_j} \psi \rangle_{\mathfrak{w}}\right)_{i,j=1}^{n}$ breaks into a product of determinants with entries so that $|x_i - x_{i+1}| \leq 1$. Hence, we are led to the case $x_i = \mathsf{x}, x_{i+1} = \mathsf{x}+1, \cdots$, which is the same as considering whether the determinant of the tridiagonal matrix $\{A_{i,j}\}_{i,j=\mathsf{x}}^{\mathsf{x}+m}$ with entries, for some $m \leq n$,

$$A_{i,j} = \delta_{i,j} + a\lambda_{j-1}\delta_{i,j-1} - a(\mu_j + \lambda_j)\delta_{i,j} + a\mu_{j+1}\delta_{i,j+1}$$

is positive. In order to answer this, we recall the following nice property of tridiagonal matrices (see page 5 of [69]): If each diagonal entry is greater than or equal to the sum of the off-diagonal entries in that row then, all its principal minors are non-negative. So, it suffices to find conditions on a such that,

$$A_{i,i} \geq A_{i,i-1} + A_{i,i+1},$$

or more explicitly,

$$1 - a(\mu_i + \lambda_i) \geq a\mu_i + a\lambda_i.$$

So we need,

$$a \leq \frac{1}{2}(\lambda_i + \mu_i)^{-1}, \forall i.$$

Thus, by letting $C = \sup_{i \geq 0} (\lambda_i + \mu_i)$ we have proven that:

Lemma 5.62. *If $a \leq \frac{1}{2C}$ then, $\mathfrak{P}_{n,n+1}^{\psi_a}$ is positive.*

Remark 5.63. *We note here, the close connection between the condition $a \leq \frac{1}{2C}$ and the true interval of orthogonality. Namely, if the support of the measure $\mathfrak{w}$ is given by $\operatorname{supp}(\mathfrak{w}) = [I^-, I^+]$, with $0 \leq I^- < I^+ \leq \infty$, then Theorem 14 of [157] gives that (c_n therein is equal to, in our notation, $\mu_n + \lambda_n$),*

$$\frac{1}{2}\left(I^- + I^+\right) \leq \limsup_{n \to \infty} \{\lambda_n + \mu_n\}$$

and thus,

$$I^+ \leq 2\limsup_{n \to \infty} \{\lambda_n + \mu_n\} \leq 2C.$$

In particular, since $2C \leq \frac{1}{a}$ the root of $\psi_a(z) = 1 - az$ is not in $[I^-, I^+]$.

Moreover, with analogous considerations if we let $\hat{C} = \sup_{i \geq 0} \left(\hat{\lambda}_i + \hat{\mu}_i\right)$ we obtain the following lemma:

Lemma 5.64. *If $b \leq \frac{1}{2\hat{C}}$ then $\mathfrak{P}_{n,n}^{\psi_b}$ is positive.*

Finally, from Lemma 5.62 and Lemma 5.64 and Proposition 5.56 we obtain as a corollary the positivity of the coherent measures:

Corollary 5.65. *Assume $\mathfrak{I}$ is compact and let $a \le \frac{1}{2C}, b \le \frac{1}{2\hat{C}}$ then, $\mathcal{M}^{\psi_a}_{n,n+1}$ and $\mathcal{M}^{\psi_b}_{n,n}$ are positive.*

5.10 Correlation kernels

5.10.1 Computation of the correlation kernel

In this subsection we assume that $\mathrm{supp}(\mathfrak{w}) = \mathfrak{I}$ is compact and that ψ is of the form,

$$\psi(x) = \psi_{t,\vec{\alpha}}(x) = \prod_{i=1}^{\mathfrak{N}} (1-\alpha_i x)e^{-tx}, \tag{5.91}$$

for some $\mathfrak{N} \in \mathbb{N}$ and $\frac{1}{2}\left(\frac{1}{C} \wedge \frac{1}{\hat{C}}\right) \ge \alpha_1 \ge \alpha_2 \ge \cdots \ge 0$ and $t \ge 0$. We denote by $\mathbb{GT}_{\mathbf{s}}(\infty)$ the set of all infinite symplectic Gelfand-Tsetlin patterns, namely infinite interlacing sequences of the following form:

$$\mathbb{GT}_{\mathbf{s}}(\infty) = \left\{\mathbb{X} = \left(\mathbb{X}^{(0,1)}, \mathbb{X}^{(1,1)}, \mathbb{X}^{(1,2)}, \cdots\right) : \mathbb{X}^{(i-1,i)} \in W^{i,i}\left(\mathbb{X}^{(i,i)}\right), \mathbb{X}^{(i,i)} \in W^{i,i+1}\left(\mathbb{X}^{(i,i+1)}\right)\right\}.$$

Define for $n \in \mathbb{N}$, the following cylinder sets $\mathfrak{C}_{n,n}\left(\mathfrak{x}^{(0,1)}, \cdots, \mathfrak{x}^{(n,n)}\right), \mathfrak{C}_{n,n+1}\left(\mathfrak{x}^{(0,1)}, \cdots, \mathfrak{x}^{(n,n+1)}\right)$ in $\mathbb{GT}_{\mathbf{s}}(\infty)$, given by,

$$\mathfrak{C}_{n,n}\left(\mathfrak{x}^{(0,1)}, \cdots, \mathfrak{x}^{(n,n)}\right) = \left\{\mathbb{X} \in \mathbb{GT}_{\mathbf{s}}(\infty) : \mathbb{X}^{(0,1)} = \mathfrak{x}^{(0,1)}, \cdots, \mathbb{X}^{(n,n)} = \mathfrak{x}^{(n,n)}\right\},$$
$$\mathfrak{C}_{n,n+1}\left(\mathfrak{x}^{(0,1)}, \cdots, \mathfrak{x}^{(n,n+1)}\right) = \left\{\mathbb{X} \in \mathbb{GT}_{\mathbf{s}}(\infty) : \mathbb{X}^{(0,1)} = \mathfrak{x}^{(0,1)}, \cdots, \mathbb{X}^{(n,n+1)} = \mathfrak{x}^{(n,n+1)}\right\}.$$

We consider the random variable X^{ψ}, taking values in $\mathbb{GT}_{\mathbf{s}}(\infty)$, with distribution Ξ^{ψ} defined by its values on the cylinder sets as follows,

$$\Xi^{\psi}\left[\mathfrak{C}_{n,n}\left(\mathfrak{x}^{(0,1)},\cdots,\mathfrak{x}^{(n,n)}\right)\right]=\mathcal{M}^{\psi}_{n,n}\left(\mathfrak{x}^{(n,n)}\right)\Lambda^{h_{n-1,n}}_{n,n}\left(\mathfrak{x}^{(n,n)},\mathfrak{x}^{(n-1,n)}\right)\times\cdots\times\Lambda^{h_{0,1}}_{1,1}\left(\mathfrak{x}^{(1,1)},\mathfrak{x}^{(0,1)}\right)$$
$$=\prod_{k=1}^{n-1}\det\left(\phi(\mathfrak{x}^{(k-1,k)}_i,\mathfrak{x}^{(k,k)}_i)\right)^{k}_{i,j=1}\det\left(\hat{\phi}(\mathfrak{x}^{(k,k)}_i,\mathfrak{x}^{(k,k+1)}_i)\right)^{k+1}_{i,j=1}$$
$$\times\det\left(\phi(\mathfrak{x}^{(n-1,n)}_i,\mathfrak{x}^{(n,n)}_i)\right)^{n}_{i,j=1}\frac{(-1)^{\binom{n-1}{2}}}{\lambda_0^{\binom{n-1}{2}}}\det\left(\langle\hat{\pi}_{\mathfrak{x}^{(n,n)}_i}\hat{Q}_{\mathfrak{x}^{(n,n)}_i},(-x)^{n-j}\psi\rangle_{\hat{\mathfrak{w}}}\right)^{n}_{i,j=1}, \tag{5.92}$$

$$\Xi^{\psi}\left[\mathfrak{C}_{n,n+1}\left(\mathfrak{x}^{(0,1)},\cdots,\mathfrak{x}^{(n,n+1)}\right)\right]=\mathcal{M}^{\psi}_{n,n+1}\left(\mathfrak{x}^{(n,n+1)}\right)\Lambda^{h_{n,n}}_{n,n+1}\left(\mathfrak{x}^{(n,n+1)},\mathfrak{x}^{(n,n)}\right)\times\cdots\times\Lambda^{h_{0,1}}_{1,1}\left(\mathfrak{x}^{(1,1)},\mathfrak{x}^{(0,1)}\right)$$
$$=\prod_{k=1}^{n}\det\left(\phi(\mathfrak{x}^{(k-1,k)}_i,\mathfrak{x}^{(k,k)}_i)\right)^{k}_{i,j=1}\det\left(\hat{\phi}(\mathfrak{x}^{(k,k)}_i,\mathfrak{x}^{(k,k+1)}_i)\right)^{k+1}_{i,j=1}$$
$$\times\frac{(-1)^{\binom{n}{2}}}{\lambda_0^{\binom{n}{2}}}\det\left(\langle\pi_{\mathfrak{x}^{(n,n+1)}_i}Q_{\mathfrak{x}^{(n,n+1)}_i},(-x)^{n+1-j}\psi\rangle_{\mathfrak{w}}\right)^{n+1}_{i,j=1}. \tag{5.93}$$

Note that, X^{ψ} is well defined by the coherency property of Proposition 5.53 and positivity of Corollary 5.65. Moreover, observe that for $\psi(x)=\psi_{t,\vec{0}}(x)=\phi_t(x)=e^{-tx}$ then (see Proposition 5.30 and the discussion following it), Ξ^{ϕ_t} gives the distribution at time t of $\mathcal{D}$-chains on odd levels and $\hat{\mathcal{D}}$-chains on even levels in $\mathbb{GT}_{\mathbf{s}}(\infty)$ interacting via the push-block dynamics, started from the fully packed initial condition.

Equivalently, we can view X^{ψ} as a random point configuration in $\mathbb{N}\times\mathbb{N}$, so that Ξ^{ψ} determines a probability measure on $2^{\mathbb{N}\times\mathbb{N}}$. Abusing notation, we will also denote this by Ξ^{ψ}. Our goal, is to calculate explicitly the correlation functions (defined below) $\{\rho^{\psi}_k\}_{k\ge 0}$ of this point process in Theorem 5.69. As above, we will denote by $(n_1,n_2)\in\{(n,n),(n,n+1)\}$ the levels of $\mathbb{GT}_{\mathbf{s}}(\infty)$. For example, $(0,1)$ denotes the first level, $(1,1)$ the second level, $(1,2)$ the third level and so on. For a point z of the form $((n_1,n_2),x)$ with (n_1,n_2) as above and $x\in\mathbb{N}$ we will say that $z\in\mathsf{X}^{\psi}$, if z belongs to the point configuration corresponding to X^{ψ}.

In what follows, we will denote by $\mathsf{C}(\mathfrak{I})$, a positively oriented (counter-clockwise) loop around $[0,I^+]$ (and *not just* around $\mathfrak{I}=[I^-,I^+]$, recall $I^-\ge 0$) that is chosen in such a way that it contains *no zeros* of ψ. Observe that, this is always possible by Remark 5.63. Our method of proof is essentially an application (of a variant) of the famous Eynard-Mehta theorem (see [35]).

We begin with some technical preliminaries but first a comment on notations. In all that follows, all the real weighted integrals over the interval $\mathfrak{I}$, for which we use the notation $\langle\cdot,\cdot\rangle_{\mathsf{m}}$, will be in the x-variable, while all the contour integrals over $\mathsf{C}(\mathfrak{I})$ will be in the variable u.

Lemma 5.66. *We have the following contour integral expressions for alternating convolutions of ϕ and $\hat{\phi}$. In the 1^{st} and 3^{rd} equalities below we have a total of $2n$ terms in the convolutions, in the*

2^{nd} *a total of* $2n+1$ *terms and in the* 4^{th} *one* $2n-1$ *terms.*

$$\left(\phi * \frac{\hat{\phi}}{\lambda_0} * \cdots * \phi * \frac{\hat{\phi}}{\lambda_0}\right)(i,j) = -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{Q_j(u)}{x-u} \rangle_{\mathfrak{w}} \frac{1}{u^n} du,$$
$$\left(\phi * \frac{\hat{\phi}}{\lambda_0} * \cdots * \frac{\hat{\phi}}{\lambda_0} * \phi\right)(i,j) = -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{\hat{Q}_j(u)}{x-u} \rangle_{\mathfrak{w}} \frac{1}{u^n} du,$$
$$\left(\frac{\hat{\phi}}{\lambda_0} * \phi * \cdots * \frac{\hat{\phi}}{\lambda_0} * \phi\right)(i,j) = -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \frac{\hat{Q}_j(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u^n} du,$$
$$\left(\frac{\hat{\phi}}{\lambda_0} * \phi * \cdots * \phi * \frac{\hat{\phi}}{\lambda_0}\right)(i,j) = -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \frac{Q_j(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u^n} du.$$

Proof. We begin by writing,

$$\begin{aligned}
\phi(i,j) = \pi_i \mathbf{1}(i \le j) = \pi_i \langle Q_i, \sum_{k=0}^{j} \pi_k Q_k \rangle_{\mathfrak{w}} = \langle \pi_i Q_i, \hat{Q}_j \rangle_{\mathfrak{w}} \\
= -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{\hat{Q}_j(u)}{x-u} \rangle_{\mathfrak{w}} du
\end{aligned}$$

and in a similar fashion,

$$\begin{aligned}
\hat{\phi}(i,j) = -\pi_i \mathbf{1}(i < j) = -\hat{\pi}_i \langle \hat{Q}_i, \sum_{k=0}^{j-1} \hat{\pi}_k \hat{Q}_k \rangle_{\hat{\mathfrak{w}}} = \langle \hat{\pi}_i \hat{Q}_i, \frac{\lambda_0}{x}(Q_j(x) - 1) \rangle_{\hat{\mathfrak{w}}} \\
= -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \lambda_0 \frac{Q_j(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u} du.
\end{aligned}$$

The last equality holds because,

$$\frac{Q_j(x) - 1}{x} = -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \frac{Q_j(u)}{u(x-u)} du \quad \text{for } x \in [0, I^+].$$

Moreover,

$$\begin{aligned}
(\phi * \hat{\phi})(i,j) &= -\frac{1}{2\pi\mathsf{i}} \sum_{k \ge 0} -\hat{\pi}_k \mathbf{1}(k < j) \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{\hat{Q}_k(u)}{x-u} \rangle_{\mathfrak{w}} du \\
&= -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, -\frac{\sum_{k=0}^{j-1} \hat{\pi}_k \hat{Q}_k(u)}{x-u} \rangle_{\mathfrak{w}} du \\
&= -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{\lambda_0}{u} \frac{Q_j(u) - 1}{x-u} \rangle_{\mathfrak{w}} du \\
&= -\frac{1}{2\pi\mathsf{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \lambda_0 \frac{Q_j(u)}{x-u} \rangle_{\mathfrak{w}} \frac{1}{u} du,
\end{aligned}$$

where the last equality follows from the fact that for all $n \geq 1$,

$$\oint_{\mathsf{C}(\mathfrak{I})} \frac{1}{u^n(x-u)} du = 0 \text{ for } x \in [0, I^+].$$

Similarly,

$$\begin{aligned}
(\hat{\phi} * \phi)(i,j) &= -\frac{1}{2\pi \mathrm{i}} \sum_{k \geq 0} \pi_k \mathbf{1}(k \leq j) \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \lambda_0 \frac{Q_k(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u} du \\
&= -\frac{1}{2\pi \mathrm{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \lambda_0 \frac{\sum_{k=0}^{j} \pi_k Q_k(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u} du \\
&= -\frac{1}{2\pi \mathrm{i}} \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \lambda_0 \frac{\hat{Q}_j(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u} du.
\end{aligned}$$

By induction, we easily obtain the statement of the lemma. □

We now define the following functions $\Phi^{(n_1,n_2)}_{(k_1,k_2)}(\cdot,\cdot)$, that will come up in the computation of the correlation kernel, on $\mathbb{N} \times \mathbb{N}$ for $(n_1, n_2) \geq (k_1, k_2)$ given by the convolutions in the Lemma above, but with $\frac{\hat{\phi}}{\lambda_0}$ replaced by $-\frac{\hat{\phi}}{\lambda_0}$ (we just put the factors $(-1)^{\binom{n}{2}}$ and $(-1)^{\binom{n-1}{2}}$ from the cylinder set distributions in the $\hat{\phi}$'s). More explicitly, we define,

$$\begin{aligned}
\Phi^{(n,n+1)}_{(k,k+1)}(i,j) &= \left(\phi * \left(-\frac{\hat{\phi}}{\lambda_0}\right) * \cdots * \phi * \left(-\frac{\hat{\phi}}{\lambda_0}\right)\right)(i,j) = (-1)^{n-k}\left(-\frac{1}{2\pi \mathrm{i}}\right) \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{Q_j(u)}{x-u} \rangle_{\mathfrak{w}} \frac{1}{u^{n-k}} du, \\
\Phi^{(n,n+1)}_{(k,k)}(i,j) &= \left(\left(-\frac{\hat{\phi}}{\lambda_0}\right) * \phi * \cdots * \phi * \left(-\frac{\hat{\phi}}{\lambda_0}\right)\right)(i,j) = (-1)^{n+1-k}\left(-\frac{1}{2\pi \mathrm{i}}\right) \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \frac{Q_j(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u^{n+1-k}} du, \\
\Phi^{(n,n)}_{(k,k)}(i,j) &= \left(\left(-\frac{\hat{\phi}}{\lambda_0}\right) * \phi * \cdots * \left(-\frac{\hat{\phi}}{\lambda_0}\right) * \phi\right)(i,j) = (-1)^{n-k}\left(-\frac{1}{2\pi \mathrm{i}}\right) \oint_{\mathsf{C}(\mathfrak{I})} \langle \hat{\pi}_i \hat{Q}_i, \frac{\hat{Q}_j(u)}{x-u} \rangle_{\hat{\mathfrak{w}}} \frac{1}{u^{n-k}} du, \\
\Phi^{(n,n)}_{(k-1,k)}(i,j) &= \left(\phi * \left(-\frac{\hat{\phi}}{\lambda_0}\right) * \cdots * \left(-\frac{\hat{\phi}}{\lambda_0}\right) * \phi\right)(i,j) = (-1)^{n-k}\left(-\frac{1}{2\pi \mathrm{i}}\right) \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{\hat{Q}_j(u)}{x-u} \rangle_{\mathfrak{w}} \frac{1}{u^{n-k}} du
\end{aligned}$$

and note that, when $(n_1, n_2) = (k_1, k_2)$ then,

$$\begin{aligned}
\Phi^{(n,n+1)}_{(n,n+1)}(i,j) &= \delta_{i,j}, \\
\Phi^{(n,n)}_{(n,n)}(i,j) &= \delta_{i,j}.
\end{aligned}$$

Moving on, for ψ as in (5.91) we define the following functions for $n, j, i \in \mathbb{N}$,

$$\mathcal{E}^{n,n+1}_{n+1-j}(i) = -\frac{1}{2\pi \mathrm{i}} \oint_{\mathsf{C}(\mathfrak{I})} \frac{Q_i(u)}{\psi(u)(-u)^{n+1-j+1}} du, \tag{5.94}$$

$$\mathcal{E}^{n,n}_{n-j}(i) = -\frac{1}{2\pi \mathrm{i}} \oint_{\mathsf{C}(\mathfrak{I})} \frac{\hat{Q}_i(u)}{\psi(u)(-u)^{n-j+1}} du. \tag{5.95}$$

Then, we have the following *biorthogonality* relations between the Ψ's and $\mathcal{E}$'s as functions of $i \in \mathbb{N}$.

Lemma 5.67.

$$\sum_{i\geq 0} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{n,n+1}_{n+1-l}(i) = \delta_{k,l}, \text{ for } k,l \leq n+1,$$
$$\sum_{i\geq 0} \Psi^{n,n}_{n-k}(i)\mathcal{E}^{n,n}_{n-l}(i) = \delta_{k,l}, \text{ for } k,l \leq n.$$

Proof. We only prove the first equality, as the second is entirely analogous,

$$\begin{aligned}\sum_{i\geq 0} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{n,n+1}_{n+1-l}(i) &= -\frac{1}{2\pi \mathrm{i}}\sum_{i\geq 0}\langle \pi_i Q_i, (-x)^{n+1-k}\psi\rangle_{\mathfrak{w}} \oint_{\mathsf{C}(\mathfrak{I})} \frac{Q_i(u)}{\psi(u)(-u)^{n+1-l+1}}du \\ &= -\frac{1}{2\pi \mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\sum_{i\geq 0}\langle \pi_i Q_i, (-x)^{n+1-k}\psi\rangle_{\mathfrak{w}} \frac{Q_i(u)}{\psi(u)(-u)^{n+1-l+1}}du \\ &= -\frac{1}{2\pi \mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})} \frac{1}{(-u)^{k-l+1}}du = \delta_{k,l}.\end{aligned}$$

□

The last technical ingredient that we need is:

Lemma 5.68. *For all $n \in \mathbb{N}$, the functions $\mathcal{E}^{n,n+1}_1(\cdot), \cdots, \mathcal{E}^{n,n+1}_{n+1}(\cdot)$ form a basis for the linear span of the functions $\left(\hat{\phi} * \Phi^{(n,n+1)}_{(0,1)}\right)(\mathrm{virt},\cdot), \cdots, \left(\hat{\phi} * \Phi^{(n,n+1)}_{(n,n+1)}\right)(\mathrm{virt},\cdot)$ and similarly $\mathcal{E}^{n,n}_1(\cdot), \cdots, \mathcal{E}^{n,n}_n(\cdot)$ form a basis for the linear span of $\left(\hat{\phi} * \Phi^{(n,n)}_{(0,1)}\right)(\mathrm{virt},\cdot), \cdots, \left(\hat{\phi} * \Phi^{(n,n)}_{(n-1,n)}\right)(\mathrm{virt},\cdot)$.*

Proof. Write $Q_i(x) = \sum_{k=0}^{i} a_k(i)x^k$. By using residue calculus and moreover since we only have a singularity at 0, we obtain that,

$$\begin{aligned}\mathcal{E}^{n,n+1}_{n+1-j}(i) &= -\frac{1}{2\pi \mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})} \frac{Q_i(u)}{\psi(u)(-u)^{n+1-j+1}}du = -\frac{(-1)^{n+1-j+1}}{(n+1-j)!}\frac{d^{n+1-j}}{du^{n+1-j}}\left(\frac{Q_i(u)}{\psi(u)}\right)\Bigg|_{u=0} \\ &= -\frac{(-1)^{n+1-j+1}}{(n+1-j)!}\sum_{l=0}^{n+1-j} f^{n+1-j}_l \frac{d^l}{du^l}Q_i(u)\Big|_{u=0} \\ &= \sum_{l=0}^{n+1-j}\tilde{f}^{n+1-j}_l a_l(i),\end{aligned}$$

where the coefficients $\{f^{n+1-j}_l\}_{l=1}^{n+1-j}$ only depend on the derivatives of $1/\psi(u)$ at $u=0$. In particular $f^{n+1-j}_{n+1-j} = \frac{1}{\psi(0)} = 1 \neq 0$ and hence also the leading coefficient $\tilde{f}^{n+1-j}_{n+1-j} \neq 0$. Thus we have,

$$span\{\mathcal{E}^{n,n+1}_1(\cdot), \cdots, \mathcal{E}^{n,n+1}_{n+1}(\cdot)\} = span\{a_0(\cdot), \cdots, a_n(\cdot)\}.$$

Similarly, if we write $\hat{Q}_i(x) = \sum_{k=0}^{i} \hat{a}_k(i)x^k$ then,

$$\mathcal{E}_{n-j}^{n,n}(i) = \sum_{l=0}^{n-j} \tilde{g}_l^{n-j} \hat{a}_l(i),$$

with $\tilde{g}_{n-j}^{n-j} \neq 0$. Hence,

$$span\{\mathcal{E}_1^{n,n}(\cdot), \cdots, \mathcal{E}_n^{n,n}(\cdot)\} = span\{\hat{a}_0(\cdot), \cdots, \hat{a}_{n-1}(\cdot)\}.$$

On the other hand, for $0 \leq k \leq n$, we have that,

$$\begin{aligned}
\left(\hat{\phi} * \Phi_{(k,k+1)}^{(n,n+1)}\right)(\mathrm{virt}, j) &= \sum_{i\geq 0} (-1)^{n-k} \left(-\frac{1}{2\pi \mathrm{i}}\right) \oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{Q_j(u)}{x-u} \rangle_{\mathfrak{w}} \frac{1}{u^{n-k}} du \\
&= (-1)^{n-k} \left(-\frac{1}{2\pi \mathrm{i}}\right) \oint_{\mathsf{C}(\mathfrak{I})} \frac{Q_j(u)}{-u} \frac{1}{u^{n-k}} du \\
&= (-1)^{n-k} a_{n-k}(j).
\end{aligned}$$

Hence,

$$span\{\left(\hat{\phi} * \Phi_{(0,1)}^{(n,n+1)}\right)(\mathrm{virt}, \cdot), \cdots, \left(\hat{\phi} * \Phi_{(n,n+1)}^{(n,n+1)}\right)(\mathrm{virt}, \cdot)\} = span\{a_0(\cdot), \cdots, a_n(\cdot)\}.$$

Similarly, for $1 \leq k \leq n$, we have,

$$\left(\hat{\phi} * \Phi_{(k-1,k)}^{(n,n)}\right)(\mathrm{virt}, j) = \mathrm{const}_{n,k} a_{n-k}(j)$$

and thus,

$$span\{\left(\hat{\phi} * \Phi_{(0,1)}^{(n,n)}\right)(\mathrm{virt}, \cdot), \cdots, \left(\hat{\phi} * \Phi_{(n-1,n)}^{(n,n)}\right)(\mathrm{virt}, \cdot)\} = span\{\hat{a}_0(\cdot), \cdots, \hat{a}_{n-1}(\cdot)\}.$$

The statement of the lemma is now evident. $\square$

We finally arrive at our main result, that Ξ^ψ is a determinantal point process with an explicit kernel given in terms of the orthogonal polynomials $\{Q_i\}_{i\geq 0}$, $\{\hat{Q}_i\}_{i\geq 0}$ and the spectral measures $\mathfrak{w}, \hat{\mathfrak{w}}$.

Theorem 5.69. *Let $\mathfrak{I}$ be compact and ψ be of the form (5.91). Then, the correlation functions $\{\rho_k^\psi\}_{k\geq 0}$ of Ξ^ψ are determinantal,*

$$\rho_k^\psi(z_1, \cdots, z_k) \stackrel{\mathrm{def}}{=} \Xi^\psi(\{E \in \mathbb{GT}_{\mathbf{s}}(\infty) \text{ s.t. } \{z_1, \cdots, z_k\} \subset E\}) = \det\left(\mathcal{K}^\psi(z_i, z_j)\right)_{i,j=1}^k \tag{5.96}$$

where $\mathcal{K}^{\psi}$ is given by,

$$\mathcal{K}^{\psi}\left(((n_1,n_2),i),(m_1,m_2),j)\right) = \frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})} \tilde{\mathcal{P}}_j(u)\langle \bar{\mathcal{P}}_i(x), \frac{x^{n_2}}{u^{m_2}}\frac{\psi(x)}{(x-u)\psi(u)}\rangle_{\mathfrak{m}} du$$
$$+\mathbf{1}\left((n_1,n_2)\geq(m_1,m_2)\right)\langle\bar{\mathcal{P}}_i(x), x^{n_2-m_2}\tilde{\mathcal{P}}_j(x)\rangle_{\mathfrak{m}} \tag{5.97}$$

and,

$$(\bar{\mathcal{P}},\tilde{\mathcal{P}},\mathfrak{m}) = \begin{cases}(\pi_i Q_i, Q_j, \mathfrak{w}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n+1),(m,m+1)\\ (\pi_i Q_i, \hat{Q}_j, \mathfrak{w}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n+1),(m,m)\\ (\hat{\pi}_i \hat{Q}_i, Q_j, \hat{\mathfrak{w}}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n),(m,m+1)\\ (\hat{\pi}_i \hat{Q}_i, \hat{Q}_j, \hat{\mathfrak{w}}) \text{ if } (n_1,n_2),(m_1,m_2) = (n,n),(m,m)\end{cases}. \tag{5.98}$$

Proof. This is an application of a variant of the Eynard-Mehta Theorem, more specifically Proposition A.2 of [46]. Identifying the functions therein from Lemma 5.67 and Lemma 5.68 we get that,

$$\mathcal{K}^{\psi}\left(((n_1,n_2),i),(m_1,m_2),j)\right) = -\Phi_{(n_1,n_2)}^{(m_1,m_2)}(i,j)\mathbf{1}\left((n_1,n_2)<(m_1,m_2)\right) + \sum_{k=1}^{m_2}\Psi_{n_2-k}^{n_1,n_2}(i)\mathcal{E}_{m_2-k}^{m_1,m_2}(j). \tag{5.99}$$

So, we need to calculate $\sum_{k=1}^{m_2}\Psi_{n_2-k}^{n_1,n_2}(i)\mathcal{E}_{m_2-k}^{m_1,m_2}(j)$. The calculation of this sum is elementary but rather tedious. Moreover, all the sums that are encountered in the sequel are finite, so there are no further issues with convergence other than the ones encountered already. We can assume $(n_1,n_2)=(n,n+1),(m_1,m_2)=(m,m+1)$, as all other cases are analogous; we just need to change Q_i's to $\hat{Q}_i$'s and $\mathfrak{w}$ to $\hat{\mathfrak{w}}$, note that in particular we are not using any specific properties of the Q_i's or $\mathfrak{w}$ below.

We first assume that $m\leq n$. Then (note that, for $k\leq m+1$ we have $\mathsf{R}_{k-(n+1)}^{\psi}=\psi$),

$$\sum_{k=1}^{m+1}\Psi_{n+1-k}^{n,n+1}(i)\mathcal{E}_{m+1-k}^{m,m+1}(j) = -\frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{m+1}\langle\pi_i Q_i,(-x)^{n+1-k}\psi\rangle_{\mathfrak{w}}\oint_{\mathsf{C}(\mathfrak{I})}\frac{Q_j(u)}{\psi(u)(-u)^{m+2-k}}du$$
$$= -\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_i Q_i,\sum_{k=1}^{m+1}\frac{(-x)^{n+1-k}}{(-u)^{m+2-k}}\psi\rangle_{\mathfrak{w}}\frac{Q_j(u)}{\psi(u)}du.$$

By using,

$$\sum_{k=1}^{m+1}\frac{(-x)^{n+1-k}}{(-u)^{m+2-k}} = \frac{u}{x-u}\left(1-\left(\frac{u}{x}\right)^{m+1}\right)\frac{(-x)^{n+1}}{(-u)^{m+2}},$$

we get,

$$\sum_{k=1}^{m+1} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j) = -\frac{1}{2\pi\mathsf{i}}\oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{u}{x-u}\left(1-\left(\frac{u}{x}\right)^{m+1}\right)\frac{(-x)^{n+1}}{(-u)^{m+2}}\psi\rangle_{\mathfrak{w}} \frac{Q_j(u)}{\psi(u)}du$$
$$= \frac{1}{2\pi\mathsf{i}}\oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{1}{x-u}\frac{(-x)^{n+1}}{(-u)^{m+1}}\psi\rangle_{\mathfrak{w}} \frac{Q_j(u)}{\psi(u)}du + \langle \pi_i Q_i, (-x)^{(n+1)-(m+1)}Q_j\rangle_{\mathfrak{w}},$$

where we have taken the residue at $u = x$ in the second term.

We now assume that $m \geq n+1$. We split the sum into two,

$$\sum_{k=1}^{m+1} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j) = \sum_{k=1}^{n+1} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j) + \sum_{k=n+2}^{m+1} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j). \tag{5.100}$$

We calculate the first summand as before,

$$\sum_{k=1}^{n+1} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j) = -\frac{1}{2\pi\mathsf{i}}\oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \frac{u}{x-u}\left(1-\left(\frac{u}{x}\right)^{n+1}\right)\frac{(-x)^{n+1}}{(-u)^{m+2}}\psi\rangle_{\mathfrak{w}} \frac{Q_j(u)}{\psi(u)}du. \tag{5.101}$$

For the second summand first recall that $\Psi^{n,n+1}_{n+1-k}(i) = \langle \pi_i Q_i, (-x)^{n+1-k}\mathsf{R}^{\psi}_{k-(n+1)}\rangle_{\mathfrak{w}}$ where $\mathsf{R}^{\psi}_{k-(n+1)}(x) = \psi(x) - \sum_{l=0}^{k-(n+1)-1} \frac{\psi^{(l)}(0)}{l!}(-x)^l(-1)^l$ and thus,

$$\sum_{k=n+2}^{m+1} \Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j) = -\frac{1}{2\pi\mathsf{i}}\oint_{\mathsf{C}(\mathfrak{I})} \langle \pi_i Q_i, \sum_{k=n+2}^{m+1}\frac{(-x)^{n+1-k}}{(-u)^{m+2-k}}\left[\psi(x) - \sum_{l=0}^{k-(n+1)-1}\frac{\psi^{(l)}(0)}{l!}(-x)^l(-1)^l\right]\rangle_{\mathfrak{w}} \frac{Q_j(u)}{\psi(u)}du. \tag{5.102}$$

So, we need to calculate,

$$\sum_{k=n+2}^{m+1}\frac{(-x)^{n+1-k}}{(-u)^{m+2-k}}\left[\psi(x) - \sum_{l=0}^{k-(n+1)-1}\frac{\psi^{(l)}(0)}{l!}(-x)^l(-1)^l\right] = \frac{1}{(-u)^{m+2}}\left[\sum_{k=n+2}^{m+1}(\psi(x)-\psi(0))\frac{(-u)^k}{(-x)^{k-(n+1)}}\right.$$
$$\left. - \sum_{k=n+3}^{m+1}\sum_{l=0}^{k-(n+2)}\frac{\psi^{(l)}(0)}{l!}\frac{(-u)^k(-1)^l}{(-x)^{k-(n+1)-l}}\right].$$

Repeatedly using the geometric summation identity we get that this is equal to,

$$
\begin{aligned}
&\frac{1}{(-u)^{m+2}}\Bigg[(\psi(x)-\psi(0))\frac{(-1)(-u)^{n+2}}{x-u}\left(1-\left(\frac{u}{x}\right)^{(m+1)-(n+1)}\right)\\
&\quad-\sum_{r=1}^{(m+1)-(n+1)-1}\frac{\psi^{(r)}(0)}{r!}\frac{(-1)(-u)^{n+2+r}(-1)^r}{x-u}\left(1-\left(\frac{u}{x}\right)^{(m+1)-(n+1)-r}\right)\Bigg]\\
&=-\frac{(-u)^{(n+1)-(m+1)}}{x-u}\left[\psi(x)-\psi(0)-\sum_{r=1}^{(m+1)-(n+1)-1}\frac{\psi^{(r)}(0)}{r!}(-u)^r(-1)^r\right]\\
&\quad+\frac{(-x)^{(n+1)-(m+1)}}{x-u}\left[\psi(x)-\psi(0)-\sum_{r=1}^{(m+1)-(n+1)-1}\frac{\psi^{(r)}(0)}{r!}(-x)^r(-1)^r\right]\\
&=-\frac{(-u)^{(n+1)-(m+1)}}{x-u}\left[\mathsf{R}^{\psi}_{(m+1)-(n+1)}(u)-\psi(u)+\psi(x)\right]+\frac{(-x)^{(n+1)-(m+1)}}{x-u}\mathsf{R}^{\psi}_{(m+1)-(n+1)}(x).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{k=n+2}^{m+1}\Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j)&=\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-u)^{(n+1)-(m+1)}}{x-u}\left[\mathsf{R}^{\psi}_{(m+1)-(n+1)}(u)-\psi(u)+\psi(x)\right]\rangle_{\mathfrak{w}}\frac{Q_j(u)}{\psi(u)}du\\
&\quad-\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-x)^{(n+1)-(m+1)}}{x-u}\mathsf{R}^{\psi}_{(m+1)-(n+1)}(x)\rangle_{\mathfrak{w}}\frac{Q_j(u)}{\psi(u)}du.
\end{aligned}
$$

Now, by taking the residue at $u=x$, in both contour integrals in the terms involving $\mathsf{R}^{\psi}_{(m+1)-(n+1)}$ we get (note that there is no pole at $u=0$ in the first contour integral),

$$
\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-u)^{(n+1)-(m+1)}}{x-u}\mathsf{R}^{\psi}_{(m+1)-(n+1)}(u)\rangle_{\mathfrak{w}}\frac{Q_j(u)}{\psi(u)}du-\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-x)^{(n+1)-(m+1)}}{x-u}\mathsf{R}^{\psi}_{(m+1)-(n+1)}(x)\rangle_{\mathfrak{w}}\frac{Q_j(u)}{\psi(u)}du\\
&=-\langle\pi_iQ_i,\frac{(-x)^{(n+1)-(m+1)}}{\psi(x)}\mathsf{R}^{\psi}_{(m+1)-(n+1)}(x)Q_j(x)\rangle_{\mathfrak{w}}+\langle\pi_iQ_i,\frac{(-x)^{(n+1)-(m+1)}}{\psi(x)}\mathsf{R}^{\psi}_{(m+1)-(n+1)}(x)Q_j(x)\rangle_{\mathfrak{w}}=0.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\sum_{k=n+2}^{m+1}\Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j)=\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-u)^{(n+1)-(m+1)}}{x-u}\left[-\psi(u)+\psi(x)\right]\rangle_{\mathfrak{w}}\frac{Q_j(u)}{\psi(u)}du\\
&=-\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-u)^{(n+1)-(m+1)}}{x-u}\rangle_{\mathfrak{w}}Q_j(u)du+\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-u)^{(n+1)-(m+1)}}{x-u}\psi(x)\rangle_{\mathfrak{w}}\frac{Q_j(u)}{\psi(u)}du.
\end{aligned}
$$

Thus, combining with the first summand we get that for $m>n$,

$$
\begin{aligned}
\sum_{k=1}^{m+1}\Psi^{n,n+1}_{n+1-k}(i)\mathcal{E}^{m,m+1}_{m+1-k}(j)&=-\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-u)^{(n+1)-(m+1)}}{x-u}\rangle_{\mathfrak{w}}Q_j(u)du\\
&\quad+\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})}\langle\pi_iQ_i,\frac{(-x)^{n+1}\psi(x)}{(-u)^{m+1}(x-u)\psi(u)}\rangle_{\mathfrak{w}}Q_j(u)du. \qquad (5.103)
\end{aligned}
$$

To obtain the correlation kernel for $m>n$, recall that there is also a contribution from

$\Phi_{(n,n+1)}^{(m,m+1)}$ which is given by,

$$\Phi_{(n,n+1)}^{(m,m+1)}(i,j) = (-1)^{n-m}\left(-\frac{1}{2\pi\mathrm{i}}\right)\oint_{\mathsf{C}(\mathfrak{I})}\langle \pi_i Q_i, \frac{Q_j(u)}{x-u}\rangle_{\mathfrak{w}}\frac{1}{u^{m-n}}du$$
$$= -\frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})} Q_j(u)\langle \pi_i Q_i, \frac{(-u)^{(n+1)-(m+1)}}{x-u}\rangle_{\mathfrak{w}}du.$$

Putting it all together, we get that,

$$\mathcal{K}^{\psi}\left(((n,n+1),i),(m,m+1),j)\right) = \frac{1}{2\pi\mathrm{i}}\oint_{\mathsf{C}(\mathfrak{I})} Q_j(u)\langle \pi_i Q_i, \frac{(-x)^{n+1}}{(-u)^{m+1}}\frac{\psi(x)}{(x-u)\psi(u)}\rangle_{\mathfrak{w}}du$$
$$+\mathbf{1}\left(n \geq m\right)\langle \pi_i Q_i, (-x)^{(n+1)-(m+1)}Q_j\rangle_{\mathfrak{w}}. \quad (5.104)$$

Multiplying by the conjugating factor $(-1)^{(n+1)-(m+1)}$ (these do not alter the correlation kernel since they vanish when we take the determinant), we obtain the statement of the Theorem. □

5.10.2 Large time and finite distance from wall limit

We now take $\psi(u) = \psi_t(u) = e^{-tu}$ so that we are considering the push-block dynamics and we want to take a large time limit while zooming in and looking at particles being at a finite distance from the wall.

More precisely, let $t \sim N\tau$ and $m, n \sim N\eta$ so that moreover, the differences between the different levels $m-n$ is constant. Furthermore note, that i, j which govern the position of the particles will be fixed and not scaled with N. This of course, avoids any delicate asymptotics involving the orthogonal polynomials $Q_i, \hat{Q}_i$ or the spectral measures $\mathfrak{w}, \hat{\mathfrak{w}}$. The exact statement of the result is as follows:

Theorem 5.70. *Let* $t(N) = N\tau$ *and*

$$(\tilde{m}_1(N), \tilde{m}_2(N)) = \left(\lfloor N\eta\rfloor + m_1, \lfloor N\eta\rfloor + m_2\right),$$
$$(\tilde{n}_1(N), \tilde{n}_2(N)) = \left(\lfloor N\eta\rfloor + n_1, \lfloor N\eta\rfloor + n_2\right),$$

with $\alpha = \frac{\eta}{\tau}$. *Then we have:*

$$\lim_{N\to\infty}\mathcal{K}^{\psi_{t(N)}}\left(\left(\left(\tilde{n}_1(N), \tilde{n}_2(N)\right), i\right), \left(\tilde{m}_1(N), \tilde{m}_2(N)\right), j\right)\right) = \mathfrak{K}_{\alpha}\left(\left(\left(n_1, n_2\right), i\right), \left(m_1, m_2\right), j\right)\right)$$
$$= \int_{I^-}^{I^+}\left[-\mathbf{1}(x \geq \alpha) + \mathbf{1}\left((n_1, n_2) \geq (m_1, m_2)\right)\right]\bar{\mathcal{P}}_i(x)x^{n_2-m_2}\tilde{\mathcal{P}}_j(x)d\mathfrak{m}(x).$$

Proof. First, note that the term:

$$\mathbf{1}\left((\tilde{n}_1(N), \tilde{n}_2(N)) \geq (\tilde{m}_1(N), \tilde{m}_2(N))\right)\langle \bar{\mathcal{P}}_i(x), x^{\tilde{n}_2(N)-\tilde{m}_2(N)}\tilde{\mathcal{P}}_j(x)\rangle_{\mathfrak{m}} = \mathbf{1}\left((n_1, n_2) \geq (m_1, m_2)\right)\langle \bar{\mathcal{P}}_i(x), x^{n_2-m_2}\tilde{\mathcal{P}}_j(x)\rangle_{\mathfrak{m}}$$

remains constant in N. We hence, focus on the double integral term of the kernel $\mathcal{K}^{\psi_{t(N)}}$ and write it as (recall $\mathfrak{I} = [I^-, I^+]$),

$$\frac{1}{2\pi \mathrm{i}} \int_{I^-}^{I^+} \oint_{\mathsf{C}(\mathfrak{I})} \frac{e^{-t(N)x+\tilde{n}_2(N)log(x)}}{e^{-t(N)u+\tilde{m}_2(N)log(u)}} \frac{\tilde{\mathcal{P}}_j(u)\bar{\mathcal{P}}_i(x)}{(x-u)} d\mathfrak{m}(x)du.$$

Write the term involving exponentials as,

$$\frac{e^{-t(N)x+\tilde{n}_2(N)log(x)}}{e^{-t(N)u+\tilde{m}_2(N)log(u)}} = \frac{e^{-N(\tau x-\eta log(x))}}{e^{-N(\tau u-\eta log(u))}} + o_N(1).$$

Let $f(z) = \tau z - \eta log(z)$. Then $f'(z) = \tau - \frac{\eta}{z}$ and so $z = \alpha \stackrel{\text{def}}{=} \frac{\eta}{\tau}$ is a critical point. Write,

$$\frac{e^{-N(\tau x-\eta log(x))}}{e^{-N(\tau u-\eta log(u))}} = \frac{e^{-N(f(x)-f(\alpha))}}{e^{-N(f(u)-f(\alpha))}}.$$

We would like to deform the $\mathsf{C}(\mathfrak{I})$ contour to a contour C_s so that,

$$\begin{aligned} &\Re\left(f(x) - f(\alpha)\right) \geq 0\,, \text{ for } \ x \in [0, I^+], \\ &\Re\left(f(u) - f(\alpha)\right) < 0\,, \text{ for u on the } \mathsf{C}_\mathsf{s} \text{ contour} \end{aligned}$$

and thus, the double integral will converge uniformly to zero as $N \to \infty$. In the process however, we might pick some residues from the pole of $\frac{1}{x-u}$ depending on how α compares with I^+. First note that for $x \in \mathbb{R}$, $\Re(f(x) - f(\alpha)) \leq 0$ is equivalent to,

$$\alpha e^{\frac{x}{\alpha}-1} \geq |x|.$$

Hence, there exists $\beta < 0$ so that $\Re(f(x)-f(\alpha)) < 0$ for $x < \beta$ and $\Re(f(x)-f(\alpha)) > 0$ for $x > \beta$ except at α. Similarly, with $u = x + iy$ the inequality $\Re\left(f(u) - f(\alpha)\right) < 0$ is then equivalent to,

$$\alpha e^{\frac{x}{\alpha}-1} < (x^2 + y^2)^{\frac{1}{2}}$$

and note that $\sup_{\beta \leq x \leq \alpha} \alpha e^{\frac{x}{\alpha}-1} = \alpha$. We can thus deform the $\mathsf{C}(\mathfrak{I})$ contour to a contour C_s that is equal to a rectangle with sides parallel to the real and imaginary axes so that the two sides that are parallel to the imaginary axis have real parts $r_1 = \alpha$ and $r_2 < \beta$ and the two sides that are parallel to the real axis have imaginary parts $im_1 > \alpha$ and $im_2 < -\alpha$. Then, on this contour we have $\Re\left(f(u) - f(\alpha)\right) < 0$ except at α, where it vanishes. If $\alpha \leq I^+$ in the course of this deformation we also pick the residue at $u = x$ which gives the single integral,

$$-\int_{I^-}^{I^+} \mathbf{1}(x \geq \alpha)\bar{\mathcal{P}}_i(x)x^{n_2-m_2}\tilde{\mathcal{P}}_j(x)d\mathfrak{m}(x).$$

Thus, for $\alpha > I^+$ the kernel $\mathcal{K}^{\psi_{t(N)}}$ converges to a triangular matrix whose diagonal entries are 1. This corresponds to the frozen or fully packed region; the particles at high levels haven't had time to move yet since $\eta > \tau I^+$. On the other hand, for $\alpha \leq I^+$, in the scaling regime considered here, $\mathcal{K}^{\psi_{t(N)}}$ converges to a kernel $\mathfrak{K}_\alpha$ with entries,

$$\mathfrak{K}_\alpha\left(((n_1,n_2),i),((m_1,m_2),j)\right)=\int_{I^-}^{I^+}\left[-\mathbf{1}(x\geq\alpha)+\mathbf{1}\left((n_1,n_2)\geq(m_1,m_2)\right)\right]\bar{\mathcal{P}}_i(x)x^{n_2-m_2}\tilde{\mathcal{P}}_j(x)d\mathfrak{m}(x). \tag{5.105}$$

□

Remark 5.71. *[Multilevel extension of discrete ensembles] As already mentioned in the introduction Borodin and Olshanski in Section 3 of [30] introduced the so called discrete determinantal ensembles associated to continuous orthogonal polynomials.*

Their definition goes as follows: suppose $\mathcal{W}(dx)$ is a weight on $\mathbb{R}$ for which the moment problem is determinate (see [30] for the precise statements). Let $P_k^*(x)$ be the k^{th} orthonormal polynomial with respect to this weight with positive leading coefficient. The discrete ensemble associated to the weight $\mathcal{W}(dx)$ (or equivalently to the polynomials $P_k^*(x)$) is the determinantal point process with the following kernel $\mathsf{K}_r^{\mathcal{W}}(i,j)$:

$$\mathsf{K}_r^{\mathcal{W}}(i,j)=\int_r^\infty P_i^*(x)P_j^*(x)\mathcal{W}(dx).$$

It is easy to see that if restricted to single levels $\mathfrak{K}_\alpha\left(((n,n+1),i),(n,n+1),j)\right)$ gives rise to the determinantal ensemble with kernel $\mathsf{K}_\alpha^{\mathfrak{w}}(i,j)$ and also $\mathfrak{K}_\alpha\left(((n,n),i),(n,n),j)\right)$ gives rise to the ensemble governed by the kernel $\mathsf{K}_\alpha^{\hat{\mathfrak{w}}}(i,j)$; since conjugation by a function does not alter the correlation functions and thus the determinantal measure.

Thus, $\mathfrak{K}_\alpha\left(((n_1,n_2),i),(m_1,m_2),j)\right)$ provide a novel multilevel determinantal extension of these discrete ensembles, so that particles on consecutive levels interlace (by construction). Moreover, in this generality, it is the first time that these ensembles appear in a concrete interacting particle system.

5.11 Appendix

5.11.1 Technical results

Proof of Lemma 5.5. We will show that for $x,y\in\mathbb{Z}$ and $t\geq0$,

$$p_t(x,y)=-\bar{\nabla}_y\sum_{w=x}^{\infty}\hat{p}_t(y,w),$$

from which the statement of Lemma 5.5 follows. It will be more convenient to write this equality in matrix form. Define the doubly infinite matrices U, V as follows,

$$A_{ij} = \begin{cases} 1 & j \geq i \\ 0 & \text{otherwise} \end{cases} \quad , \quad B_{ij} = \begin{cases} 1 & i = j \\ -1 & j = i+1 \\ 0 & \text{otherwise} \end{cases} .$$

Observe that, $AB = BA = Id$ and moreover and this is the key relation, $B\mathcal{D} = \hat{\mathcal{D}}^{\mathrm{T}}B$ where $\hat{\mathcal{D}}^{\mathrm{T}}$ denotes the transpose of $\hat{\mathcal{D}}$. Then, with this notation in place we want to show,

$$P(t) = A\hat{P}^{\mathrm{T}}(t)B \stackrel{\text{def}}{=} P_*(t), \text{ for } t \geq 0.$$

First note that $P_*(0) = Id$ and moreover,where in the first equality we interchange $\frac{d}{dt}$ and an infinite sum which will be justified below, and in the second we use the backwards equation, for $t > 0$,

$$\begin{aligned} \frac{d}{dt}P_*(t) &= A\left(\frac{d}{dt}\hat{P}(t)\right)^{\mathrm{T}} B \\ &= A\left(\hat{\mathcal{D}}\hat{P}(t)\right)^{\mathrm{T}} B \\ &= A\hat{P}^{\mathrm{T}}(t)\hat{\mathcal{D}}^{T}B \\ &= A\hat{P}^{\mathrm{T}}(t)B\mathcal{D} = P_*(t)\mathcal{D}. \end{aligned}$$

Finally, note that $-\bar{\nabla}_y \sum_{w=x}^{\infty} \hat{p}_t(y,w) \geq 0$ and $\sum_{y\in\mathbb{Z}} -\bar{\nabla}_y \sum_{w=x}^{\infty} \hat{p}_t(y,w) = 1$. Hence, by uniqueness of solutions to the forwards equation we obtain that for $t \geq 0$, $P_*(t) = P(t)$. Now, in order to justify the interchange of summation and differentiation it suffices to show that the series,

$$\sum_{w=x}^{\infty} \frac{d}{dt}\hat{p}_t(y,w)$$

converges uniformly on compact intervals of t, where $x, y \in \mathbb{Z}$ are fixed. First, note that for $n \geq 1$ we have,

$$\sum_{w=x}^{x+n} \frac{d}{dt}\hat{p}_t(y,w) = \hat{\lambda}(y)\sum_{w=x}^{x+n}\hat{p}_t(y-1,w) - \left(\hat{\lambda}(y)+\hat{\mu}(y)\right)\sum_{w=x}^{x+n}\hat{p}_t(y,w) + \hat{\mu}(y)\sum_{w=x}^{x+n}\hat{p}_t(y+1,w). \tag{5.106}$$

Hence, $\sum_{w=x}^{x+n} \frac{d}{dt}\hat{p}_t(y,w)$ converges on $0 \leq t < \infty$ and moreover, has uniformly bounded partial sums. More specifically,

$$\sum_{w=x}^{x+n} \left|\frac{d}{dt}\hat{p}_t(y,w)\right| \leq 2\left(\hat{\lambda}(y)+\hat{\mu}(y)\right) , \ \forall t \geq 0, \forall n \geq 1.$$

Thus, the partial sums of,

$$\sum_{w=x}^{\infty} \hat{p}_t(y,w)$$

are uniformly bounded and equicontinuous, which can be seen as follows. If we define, for fixed $x, y \in \mathbb{Z}$, $f_n(t) = \sum_{w=x}^{x+n} \hat{p}_t(y,w)$ we obviously have $|f_n(t)| \le 1, \forall t \ge 0$ and $n \ge 1$. Moreover, for $s \le t$ in $[0,T]$ we have by the Mean Value Theorem, for some $u \in (s,t)$,

$$f_n(t) - f_n(s) = (t-s)\frac{d}{du} f_n(u)$$

and hence,

$$\begin{aligned} |f_n(t) - f_n(s)| \le \left| \sum_{w=x}^{x+n} \frac{d}{du} \hat{p}_u(x,y) \right| &\le |t-s| \sup_{u \in [0,T]} \sum_{w=x}^{x+n} \left| \frac{d}{du} \hat{p}_u(y,w) \right| \\ &\le 2\left(\hat{\lambda}(y) + \hat{\mu}(y)\right)|t-s|\,, \ \forall n \ge 1. \end{aligned}$$

So, by the Arzela Ascoli Theorem we obtain that the series $\sum_{w=x}^{\infty} \hat{p}_t(y,w)$ converges uniformly on every finite interval in t and hence by equality (5.106) the series $\sum_{w=x}^{\infty} \frac{d}{dt} \hat{p}_t(y,w)$ does so as well. By iterating the same argument, we also see that this holds for $\sum_{w=x}^{\infty} \frac{d^k}{dt^k} \hat{p}_t(y,w)$ for any $k \ge 1$. □

Proof of Proposition 5.7. The result is implied from the following two claims, for $s \le t$, $x, x', x'', w \in I$:

1. If $F_{s,t}(w) = x' \le x$ then $G_{s,t}(x) \ge w$.

2. If $F_{s,t}(w) = x'' > x$ then $G_{s,t}(x) < w$.

To show the first one, observe that without loss of generality we can assume that $F_{s,t}(w) = x$. Then, *attempt* to follow the original/forwards path starting from w at time s and that ends at x at time t backwards in time, using only the *red* arrows, until the first time this is no longer possible. This happens iff the original/forwards path/chain came up using an up $\uparrow$ arrow or the chain running backwards encounters a *red* up $\uparrow$ arrow. The claim then follows, since the backwards path always stays above the original/forwards path.

To show the second one, note that without loss of generality we can assume that $F_{s,t}(w) = x + 1$. Consider the last instance (if they never meet the claim is trivial) $\tau < t$ the forwards path starting from w at time s and moving according to the original arrows and the backwards path starting from x at time t and using the *red* arrows are together. This is equivalently, the first instance (cf. right continuity) they meet, with time running backwards from t. This can only happen if the forwards path encounters an up $\uparrow$ arrow which means the backwards path encountered a down *red* $\downarrow$ arrow, which gives a contradiction. This is since the paths would split at τ, with time running backwards in such cases. □

5.11.2 Projective chains from branching of functions

Suppose we are given $\forall n \in \mathbb{N}$, indexing sets $I_n \subset \mathbb{Z}^n$, Polish spaces $\mathcal{X}^n = \mathcal{X} \times \cdots \times \mathcal{X}$, a distinguished point $\bar{u} \in \mathcal{X}$, Borel measures w_n on $\mathcal{X}^n$ and finally families of functions $\{F_n(x; u_1, \cdots, u_n)\}_{x \in I_n}$ orthogonal in $L^2(\mathcal{X}^n, w_n)$ normalized so that $F_n(x; \bar{u}, \cdots, \bar{u}) = 1$, $\forall n \in \mathbb{N}, x \in I_n$. Consider the convex set, denoted by $\mathcal{Y}_n$, consisting of functions $\mathcal{F}_n$ such that the following series converges uniformly in $\mathcal{X}^n$ (this can be relaxed) and in $L^2(\mathcal{X}^n, w_n)$,

$$\mathcal{F}_n^{M_n}(u_1, \cdots, u_n) = \sum_{x \in I_n} M_n(x) F_n(x; u_1, \cdots, u_n), \tag{5.107}$$

where,

$$M_n(x) \geq 0, \ \forall x \in I_n \text{ and } \sum_{x \in I_n} M_n(x) = 1. \tag{5.108}$$

Note that, by the orthogonality of the $\{F_n(x; \cdot)\}_{x \in I_n}$ we obtain that the $\{M_n(x)\}_{x \in I_n}$ are determined uniquely by the $\mathcal{F}_n(\cdot)$ as follows,

$$M_n(x) = \frac{\langle \mathcal{F}_n(\cdot), F_n(x; \cdot) \rangle_{w_n}}{\langle F_n(x; \cdot), F_n(x; \cdot) \rangle_{w_n}}. \tag{5.109}$$

Now, further assume that,

$$F_n(x; u_1, \cdots, u_{n-1}, \bar{u}) = \sum_{y \in I_{n-1}} \Lambda_{n-1}^n(x, y) F_{n-1}(y; u_1, \cdots, u_{n-1}), \tag{5.110}$$

for some Markov kernels, Λ_{n-1}^n from I_n to I_{n-1} i.e.

$$\Lambda_{n-1}^n(x, y) \geq 0, \ \forall x \in I_n, y \in I_{n-1} \text{ and (necessarily) } \sum_{y \in I_{n-1}} \Lambda_{n-1}^n(x, y) = 1.$$

Moreover, we assume that for any fixed $x \in I_n$ the measure $\Lambda_{n-1}^n(x, \cdot)$ is supported on *finitely many* $y \in I_{n-1}$. Observe that, this is always the case for branching graphs by definition. In particular, the functions $\{F_n(x; \cdot)\}_{x \in I_n, n \geq 1}$ generate a projective chain with levels $\{I_n\}_{n \geq 1}$ and Markov links from I_n to I_{n-1} given by $\Lambda_{n-1}^n(x, y)$ with $x \in I_n$ and $y \in I_{n-1}$.

Remark 5.72. *In the case of the alternating construction, $I_n = W^n(\mathbb{N})$, $\mathcal{X} = [0, I^+]$ and $\bar{u} = 0$. For $\nu \in I_n$ and $u_1, \cdots, u_n \in [0, I^+]$, the functions $F_n(\nu; u_1, \cdots, u_n)$ are given by (cf. (5.71)),*

$$F_n(\nu; u_1, \cdots, u_n) = \frac{h_{n-1,n}(\nu; u_1, \cdots, u_n)}{h_{n-1,n}(\nu; 0, \cdots, 0)} = \frac{h_{n-1,n}(\nu; u_1, \cdots, u_n)}{h_{n-1,n}(\nu)}$$

and the Markov kernels $\Lambda_{n-1}^n(\nu, \kappa)$, for $\nu \in W^n$ and $\kappa \in W^{n-1}$, as follows,

$$\Lambda_{n-1}^n(\nu, \kappa) = \left(\Lambda_{n-1,n}^{h_{n-1,n-1}} \Lambda_{n-1,n-1}^{h_{n-2,n-1}} \right)(\nu, \kappa).$$

Moving on to coherent measures, the fact that $M_n\Lambda^n_{n-1} = M_{n-1}$ is equivalent to,

$$\mathcal{F}_n^{M_n}(u_1, \cdots, u_{n-1}, \bar{u}) = \sum_{y \in I_{n-1}} M_{n-1}(y) F_{n-1}(y; u_1, \cdots, u_{n-1}). \tag{5.111}$$

This can be seen as follows. If $M_n\Lambda^n_{n-1} = M_{n-1}$, we multiply both sides of (5.110) by $M_n(x)$ and sum over $x \in I_n$ first (there is only one infinite sum here so we can interchange them without any issues) to arrive at (5.111). On the other hand, if (5.111) holds we can again multiply (5.110) by $M_n(x)$ and sum over $x \in I_n$ to obtain using (5.111),

$$\sum_{y \in I_{n-1}} M_{n-1}(y) F_{n-1}(y; u_1, \cdots, u_{n-1}) = \sum_{y \in I_{n-1}} \sum_{x \in I_n} M_n(x) \Lambda^n_{n-1}(x, y) F_{n-1}\left(y; u_1, \cdots, u_{n-1}\right),$$

with both series converging uniformly and in $L^2\left(\mathcal{X}^{n-1}, w_{n-1}\right)$ and by taking the inner product with $F_{n-1}(z; \cdot)$ we get,

$$M_{n-1}(z) = \sum_{x \in I_n} M_n(x) \Lambda^n_{n-1}(x, z).$$

Thus (truncated) coherent measures up to level N, namely sequences of probability measures $\{M_n\}_{n \le N}$ such that $M_n\Lambda^n_{n-1} = M_{n-1}$ for $n \le N$ are in bijection with sequences $\{\mathcal{F}_n\}_{n \le N}$ such that $\mathcal{F}_n \in \mathcal{Y}_n$ with $\mathcal{F}_n(u_1, \cdots, u_n) = \mathcal{F}_N(u_1, \cdots, u_n, \bar{u}, \cdots, \bar{u})$. Thus, if we define $\left(\mathcal{S}\mathcal{F}_n\right)(u_1, \cdots, u_{n-1}) = \mathcal{F}_n\left(u_1, \cdots, u_{n-1}, \bar{u}\right)$ which is an affine map from $\mathcal{Y}_n$ to $\mathcal{Y}_{n-1}$ and consider the projective limit,

$$\mathcal{Y} = \lim_{\leftarrow} \mathcal{Y}_n \tag{5.112}$$

consisting of functions $\mathcal{F}_\infty$ on the space $\mathcal{X}_0^\infty = (u_1, u_2, \cdots) \in \mathcal{X} \times \mathcal{X} \times \cdots$ (having only finitely many coordinates not equal to $\bar{u}$) such that,

$$\mathcal{F}_n^{\mathcal{F}_\infty}(u_1, \cdots, u_n) \stackrel{\text{def}}{=} \mathcal{F}_\infty(u_1, \cdots, u_n, \bar{u}, \bar{u}, \cdots) \in \mathcal{Y}_n \;, \forall n \in \mathbb{N}, \tag{5.113}$$

then studying the extremal coherent measures is equivalent to the study of $\mathrm{Ex}\left(\mathcal{Y}\right)$.

5.11.3 Factorization implies extremality

We now aim to prove under several assumptions that if $\mathcal{F}_\infty$ factorizes then, the corresponding coherent measure is extremal. We will reduce the problem to an application of de Finetti's theorem, following an argument which in this particular setting, as far as we know, originates with Okounkov's and Olshanski's paper [117].

We assume that, $\forall n \in \mathbb{N}$ and $x \in I_n$, the functions $F_n\left(x; u_1, \cdots, u_n\right)$ are symmetric polynomials on $[0, I^+]^n$, orthogonal with respect to a weight w_n and $\bar{u} = 0$. It will be more convenient to work on the n-dimensional torus $\mathbb{T}^n = \{(z_1, \cdots, z_n) \subset \mathbb{C} : |z_i| = 1\}$ rather than the cube. We let $\mathfrak{W}$ denote the BC_n Weyl group namely,

$$\mathfrak{W} = S(n) \ltimes \mathbb{Z}_2^n,$$

where the symmetric group $S(n)$ acts by permuting the variables and $\mathbb{Z}_2^n$ acts as follows,

$$f(z_1, \cdots, z_n) \mapsto f(z_1^{\pm 1}, \cdots, z_n^{\pm 1}).$$

We will be interested in $\mathfrak{W}$-invariant Laurent polynomials in n variables on $\mathbb{T}^n$. It is a well known fact, that the algebra of n-variable $\mathfrak{W}$-invariant Laurent polynomials can be identified with the standard algebra of symmetric polynomials in n variables (see first paragraph of Section 2 of [142] for a discussion). More concretely, under the change of variables,

$$u_i = \frac{I^+}{2}\left(1 - \frac{z_i + z_i^{-1}}{2}\right) = \mathfrak{g}(z_i),$$

we can map symmetric polynomials on the cube $[0, I^+]^n$ to $\mathfrak{W}$-invariant Laurent polynomials on $\mathbb{T}^n$ and vice versa and note that the distinguished point $\bar{u} = 0$ gets mapped to $z = 1$. We can thus, consider the corresponding $\mathfrak{W}$-invariant Laurent polynomial to $F_n(x; u_1, \cdots, u_n)$, denoted by $G_n(x; z_1, \cdots, z_n) = F_n(x; \mathfrak{g}(z_1), \cdots, \mathfrak{g}(z_n))$, orthogonal in $L^2(\mathbb{T}^n, \tilde{w}_n)$ where $\tilde{w}_n$ is obtained by the change of variables formula. Finally, we denote the corresponding convex set $\tilde{\mathcal{Y}}_n$ consisting of functions $\mathcal{G}_n(z_1, \cdots, z_n) = \mathcal{F}_n(\mathfrak{g}(z_1), \cdots, \mathfrak{g}(z_n))$ so that,

$$\mathcal{G}_n(z_1, \cdots, z_n) = \sum_{x \in I_n} M_n(x) G_n(x; z_1, \cdots, z_n), \tag{5.114}$$

$$G_n(x; z_1, \cdots, z_{n-1}, 1) = \sum_{y \in I_{n-1}} \Lambda_{n-1}^n(x, y) G_{n-1}(y; z_1, \cdots, z_{n-1}).$$

We make the following essential (and rather non-trivial to check) *positive definiteness* assumption, namely that $\forall x \in I_n$,

$$G(x; z_1, \cdots, z_n) = \sum_{\lambda_1, \cdots, \lambda_n \in \mathbb{Z}} \mathfrak{a}(x; \lambda_1, \cdots \lambda_n) z_1^{\lambda_1} \cdots z_n^{\lambda_n}, \text{ with } \mathfrak{a}(x; \lambda_1, \cdots \lambda_n) \geq 0, \forall \lambda_1, \cdots, \lambda_n \in \mathbb{Z}.$$

Note that, since $G(x; z_1, \cdots, z_n) = 1$ this implies that,

$$\sum_{\lambda_1, \cdots, \lambda_n \in \mathbb{Z}} \mathfrak{a}(x; \lambda_1, \cdots \lambda_n) = 1$$

and so by the positivity of the $\mathfrak{a}(x; \lambda_1, \cdots \lambda_n)$ for $(z_1, \cdots, z_n) \in \mathbb{T}^n$, $|G_n(x; z_1, \cdots, z_n)| \leq 1$ and in particular the series (5.114) converges uniformly. Thus, $\mathcal{G}_n$ is a continuous,normalized, positive definite, symmetric function on $\mathbb{T}^n$.

Hence, and this is the key observation, the convex set $\tilde{\mathcal{Y}}_n$ is a subset of the convex set of characteristic functions of measures on $\mathbb{Z}^n$ invariant under the action of $S(n)$. Thus, $\tilde{\mathcal{Y}} = \varprojlim \tilde{\mathcal{Y}}_n$ the set of functions $\mathcal{G}_\infty$ on $(z_1, z_2, \cdots) \in \mathbb{T}_0^\infty$ such that,

$$\mathcal{G}_n(z_1, \cdots, z_n) \stackrel{\text{def}}{=} \mathcal{G}_\infty(z_1, \cdots, z_n, 1, 1, \cdots) \in \tilde{\mathcal{Y}}_n, \forall n \in \mathbb{N}, \tag{5.115}$$

is a (convex) subset of the convex set $\mathcal{Z}$ of characteristic functions of probability measures on $\mathbb{Z}^\infty = \mathbb{Z} \times \mathbb{Z} \times \cdots$, invariant under the action of $S(\infty)$. We have thus arrived at the following result.

Proposition 5.73. *Under the assumptions above, for $\mathcal{G}_\infty \in \tilde{\mathcal{Y}}$ further assume that there exists $\mathcal{G}_1 \in \tilde{\mathcal{Y}}_1$ such that $\forall n \geq 1$,*

$$\mathcal{G}_\infty(z_1, \cdots, z_n, 1, 1, \cdots) = \prod_{i=1}^{n} \mathcal{G}_1(z_i). \tag{5.116}$$

Then, $\mathcal{G}_\infty \in \mathrm{Ex}(\tilde{\mathcal{Y}})$.

Proof. By de Finetti's theorem and the factorization property (5.116) we have $\mathcal{G}_\infty \in \mathrm{Ex}(\mathcal{Z})$. Since $\tilde{\mathcal{Y}}$ is a convex subset of $\mathcal{Z}$ we get $\mathcal{G}_\infty \in \mathrm{Ex}(\tilde{\mathcal{Y}})$. □

Remark 5.74. *We have a Markov kernel $\Lambda_n^\infty : \mathrm{Ex}(\tilde{\mathcal{Y}}) \to I_n$, defined for $\mathcal{G}_\infty \in \mathrm{Ex}(\tilde{\mathcal{Y}})$ such that (5.116) holds, that is given as follows,*

$$\Lambda_n^\infty(\mathcal{G}_1, x) \stackrel{\text{def}}{=} M_n^{\mathcal{G}_1}(x) \stackrel{\text{def}}{=} \frac{\langle \prod^n \mathcal{G}_1(\cdot), G_n(x; \cdot)\rangle_{\tilde{w}_n}}{\langle G_n(x; \cdot), G_n(x; \cdot)\rangle_{\tilde{w}_n}}. \tag{5.117}$$

Remark 5.75. *Note that, the assumptions considered in this section are satisfied in the case of general β normalized Jack (see [117]) and Jacobi (see [118]) polynomials. Checking the positive definiteness of $G_n(\nu; \cdot)$ corresponding to $F_n(\nu; \cdot) = \frac{h_{n-1,n}(\nu;\cdot)}{h_{n-1,n}(\nu)}$, cf. (5.71) which would imply the extremality of $M_n = \mathcal{M}_{n-1,n}^{\psi}$ for $\psi(x) = p(x)e^{-tx}$ where $p(x)$ is an arbitrary polynomial is in general non-trivial.*

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